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Unified Theory of Annihilation-Creation Operators for Solvable ('Discrete') Quantum Mechanics

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Abstract

The annihilation-creation operators $a^{(\pm)}$ are defined as the positive/negative frequency parts of the exact Heisenberg operator solution for the 'sinusoidal coordinate'. Thus $a^{(\pm)}$ are hermitian conjugate to each other and the relative weights of various terms in them are solely determined by the energy spectrum. This unified method applies to most of the solvable quantum mechanics of single degree of freedom including those belonging to the 'discrete' quantum mechanics.

1 Introduction

The annihilation and creation operators are probably the most basic and important tools in quantum mechanics. Modern quantum physics is almost unthinkable without them. About eighty years after its birth, the list of exactly solvable systems in quantum mechanics [1] is quite long now, including those of the so-called 'discrete' quantum mechanics [2, 3]. One natural question is that if these exactly solvable quantum mechanical systems also possess the algebraic solution method embodied in the annihilation-creation operators. We will answer the question in the affirmative and give a *unified dynamical* derivation of the annihilation-creation operators for most of the solvable quantum mechanics of single degree of freedom including those belonging to the 'discrete' quantum mechanics.

The method is quite simple and elementary. One identifies a special function $\eta(x)$ of the space-coordinate x, which undergoes 'sinusoidal motion' at the classical and quantum levels (2.10), (2.14). The latter is simply the exact Heisenberg operator solution for $\eta(x)$. The function η is the argument of the orthogonal polynomial (2.3) constituting the eigenfunctions of the system. The positive/negative frequency parts of the exact Heisenberg operator solution of the 'sinusoidal coordinate' $\eta(x)$ (2.28), (2.29) give the annihilation-creation operators. This is essentially the same recipe as used by Heisenberg for solving the harmonic oscillator in matrix mechanics. To the best of our knowledge, the 'sinusoidal coordinate' was first introduced in a rather broad sense for general (not necessarily solvable) potentials as a useful means for coherent state research by Nieto and Simmons [4].

By similarity transformation in terms of the ground state wavefunction, the results of the present paper will be translated to those of the corresponding orthogonal polynomials. In particular, the exact Heisenberg operator solution of the 'sinusoidal coordinate' corresponds to the so-called *structure relation* [5] for the orthogonal polynomials. Our method provides the unified derivation of the structure relations for the Askey-Wilson, Wilson, continuous dual Hahn, continuous Hahn and Meixner-Pollaczek polynomials [6] based on the Hamiltonian principle. These polynomials are the eigenfunctions of the various 'discrete' quantum mechanical systems [2, 3] and they are the deformations of the Jacobi, Laguerre and Hermite polynomials [7].

This paper is organised as follows. The general theory of the annihilation-creation operators is explained with typical examples; one from the ordinary quantum mechanics and two from the 'discrete' version in section two. Further explicit results are presented in section three. They include the (symmetric) Pöshl-Teller, symmetric Rosen-Morse, Morse and x^2+1/x^2 potentials on top of the five examples belonging to the 'discrete' quantum mechanics mentioned above. Section four is for a summary and comments. In Appendix **A**, the necessary and sufficient condition for the existence of the 'sinusoidal coordinate' is analysed within the context of ordinary quantum mechanics. It turns out that those potentials having the 'sinusoidal coordinate' are all *shape invariant* [8]. In Appendix **B**, interpretation of the annihilation-creation operators within the framework of shape invariance is given. It is the mechanism underlying the solvability of all systems considered in this paper. Appendix **C** gives various definitions of the orthogonal polynomials, hypergeometric functions and their q-analogues [7, 6].

2 General theory with typical examples

The purpose of the present paper is to present a *unified dynamical* theory of **annihilation-creation** operators. It is applicable to most of the *exactly solvable* quantum mechanical systems of one degree of freedom, including the so-called 'discrete' quantum mechanics which are certain deformation of the solvable quantum mechanics [2, 3]. They satisfy certain *difference equations* instead of the second order differential equations. Generalisation to the systems of many degrees of freedom will be discussed elsewhere. The restriction to the solvable quantum systems is rather trivial and inevitable, since a system is obviously solvable if it possesses explicitly defined annihilation and creation operators and any one single eigenstate to work on. Then the entire set of exact eigenstates are easily and concretely generated.

Except for the simple harmonic oscillator, which gives probably the only so far universally accepted example of the annihilation-creation operators, there are quite a wide variety of proposed annihilation and creation operators in the literature [9]. Historically most of these annihilation-creation operators are connected to the so-called *algebraic theory of coherent states*, which are usually defined as eigenstates of *annihilation operators*. Therefore, for a given potential or a quantum Hamiltonian, there could be as many coherent states as the definitions of the annihilation operators.

Our new unified definition of the annihilation-creation operators is, on the contrary, based on the dynamical properties of a special coordinate, the 'sinusoidal coordinate' shared by a class of solvable dynamical systems discussed in this paper. A quantum mechanical system with a self-adjoint Hamiltonian $\mathcal{H}$ is solvable (or solved) if the entire set of its energy eigenvalues $\{\mathcal{E}_n\}$ and the corresponding eigenvectors $\{\phi_n\}$, $n = 0, 1, \ldots$ are known:

$$\mathcal{H}\phi_n = \mathcal{E}_n\phi_n, \quad n = 0, 1, \ldots, . \tag{2.1}$$

As is well-known a quantum Hamiltonian (together with its 'discrete' analogue) has in general discrete as well as continuous spectrum. In this paper we will concentrate on the discrete energy levels only, either finite or infinite in number. Then, because of the one-dimensionality, the eigenvalues are not degenerate

$$\mathcal{E}_0 < \mathcal{E}_1 < \cdots , \tag{2.2}$$

and the eigenvectors have finite norms $||\phi_n||^2 = N_n^2 < \infty$. (When normalised vectors are

needed we denote them by adding a hat, $\{\hat{\phi}_n \stackrel{\text{def}}{=} \phi_n/N_n\}$, $||\hat{\phi}_n|| = 1$.) This is the solution in the Schrödinger picture and the eigenvectors $\{\phi_n\}$ are usually expressed as functions $\{\phi_n(x)\}$ of the space-coordinate x. For the majority of the solvable quantum systems, the n-th eigenfunction has the following general structure [1]:

$$\phi_n(x) = \phi_0(x) P_n(\eta(x)) \tag{2.3}$$

in which $\phi_0(x)$ is the *ground state* wavefunction. It has no nodes and we may choose it to be always real and positive. The second factor $P_n(\eta(x))$ is a polynomial of degree n in a real variable η. We also take $P_n(\eta)$ as real and use a convention $P_{-1}(\eta) = 0$. Reflecting the orthogonality theorem of the eigenvectors of a self-adjoint Hamiltonian, $\{P_n(\eta)\}$ form *orthogonal polynomials* with respect to a weight function (measure)

$$\phi_0(x)^2 dx \propto w(\eta) d\eta. \tag{2.4}$$

Throughout this paper we follow the definition and notation of Szegö's book [7] for the classical orthogonal polynomials and the review by Koekoek and Swarttouw [6] for the Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, which are deformations of the classical orthogonal polynomials.

There are certain exactly solvable quantum systems, for example, the one-dimensional Kepler problem, etc, for which the general form of eigenfunction (2.3) does not hold. For them, the present unified theory does not apply. See Appendix **A**.

Our main claim is that this $\eta(x)$ undergoes a '*sinusoidal motion*' under the given Hamiltonian $\mathcal{H}$, at the classical as well as quantum level, by mimicking the simple harmonic oscillator. This fact is the basis of our dynamical and unified definition of the annihilation-creation operators. To be more specific, at the classical level we have

$$\{\mathcal{H}, \{\mathcal{H}, \eta\}\} = -\eta\, R_0(\mathcal{H}) - R_{-1}(\mathcal{H}) \tag{2.5}$$

in which the canonical Poisson bracket relations are defined for the canonical coordinate x, its conjugate momentum p and for any functions $A(x,p)$ and $B(x,p)$ as:

$$\{x, p\} = 1, \quad \{x, x\} = \{p, p\} = 0, \quad \{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}. \tag{2.6}$$

The two coefficients R_0 and R_{-1} are, in general, polynomials in the Hamiltonian $\mathcal{H}$. The effect of R_{-1} is to shift the origin of $\eta(x)$ by a quantity (possibly) depending on $\mathcal{H}$. It is

convenient to introduce a shifted sinusoidal coordinate $\tilde{\eta}(x)$

$$\tilde{\eta}(x) \stackrel{\text{def}}{=} \eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H}) \qquad \Big(\Rightarrow \{\mathcal{H}, \{\mathcal{H}, \tilde{\eta}\}\} = -\tilde{\eta}\, R_0(\mathcal{H})\Big). \tag{2.7}$$

The relation (2.5) would allow to evaluate the multiple Poisson brackets of η with $\mathcal{H}$ easily:

$$\mathrm{ad}\,\mathcal{H}\,\eta \stackrel{\text{def}}{=} \{\mathcal{H}, \eta\}, \quad (\mathrm{ad}\,\mathcal{H})^2\eta = \{\mathcal{H}, \{\mathcal{H}, \eta\}\}, \quad (\mathrm{ad}\,\mathcal{H})^n\eta = \{\mathcal{H}, (\mathrm{ad}\,\mathcal{H})^{n-1}\eta\}, \tag{2.8}$$

which leads to a simple sinusoidal time-evolution:

$$\begin{aligned}\tilde{\eta}(x;t) &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\mathrm{ad}\,\mathcal{H})^n \tilde{\eta}, \\ &= -\{\mathcal{H}, \tilde{\eta}\}_0 \frac{\sin\big[t\sqrt{R_0(\mathcal{H}_0)}\,\big]}{\sqrt{R_0(\mathcal{H}_0)}} + \tilde{\eta}(x)_0 \cos\big[t\sqrt{R_0(\mathcal{H}_0)}\,\big].\end{aligned} \tag{2.9}$$

In the original variable it reads

$$\begin{aligned}\eta(x;t) = &-\{\mathcal{H}, \eta\}_0 \frac{\sin\big[t\sqrt{R_0(\mathcal{H}_0)}\,\big]}{\sqrt{R_0(\mathcal{H}_0)}} - R_{-1}(\mathcal{H}_0)/R_0(\mathcal{H}_0) \\ &+ \big(\eta(x)_0 + R_{-1}(\mathcal{H}_0)/R_0(\mathcal{H}_0)\big) \cos\big[t\sqrt{R_0(\mathcal{H}_0)}\,\big],\end{aligned} \tag{2.10}$$

in which $\eta(x)_0$ ($\tilde{\eta}(x)_0$) and $\{\mathcal{H}, \eta\}_0$ are the initial values (at $t = 0$) of these variables and $\mathcal{H}_0$ denotes the value of the Hamiltonian (the energy) for these initial data. In general, the frequency of the simple oscillation $\sqrt{R_0(\mathcal{H}_0)}$ can depend on the initial data. This is the reason why we call $\eta(x)$ the 'sinusoidal coordinate' avoiding the more appealing but misleading "harmonic coordinate".

At the quantum level with the canonical commutation relations

$$[x, p] = i, \quad [x, x] = [p, p] = 0, \qquad \hbar \equiv 1, \tag{2.11}$$

the formula corresponding to (2.5) reads

$$[\mathcal{H}, [\mathcal{H}, \eta]\,] = \eta\, R_0(\mathcal{H}) + [\mathcal{H}, \eta]\, R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \tag{2.12}$$

In other words, the multiple commutators of $\mathcal{H}$ with η form a closed algebra at level two. Here, the quantum coefficients R_0 and R_{-1} could differ from the classical ones by quantum corrections. But we use the same symbols since there is no risk of confusion. Obviously $R_1(\mathcal{H})$ is the quantum effect. As in the classical case, the multiple commutators of η with $\mathcal{H}$

$$\mathrm{ad}\,\mathcal{H}\,\eta \stackrel{\text{def}}{=} [\mathcal{H}, \eta], \quad (\mathrm{ad}\,\mathcal{H})^2\eta = [\mathcal{H}, [\mathcal{H}, \eta]\,], \quad (\mathrm{ad}\,\mathcal{H})^n\eta = [\mathcal{H}, (\mathrm{ad}\,\mathcal{H})^{n-1}\eta], \tag{2.13}$$

can be easily evaluated from (2.12). This leads to the *exact operator solution* in the Heisenberg picture:

$$
\begin{aligned}
e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}(\mathrm{ad}\,\mathcal{H})^n\eta, \\
&= [\mathcal{H},\eta(x)]\frac{e^{i\alpha_+(\mathcal{H})t}-e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H})/R_0(\mathcal{H}) \\
&\quad + \big(\eta(x)+R_{-1}(\mathcal{H})/R_0(\mathcal{H})\big)\frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t}+\alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})},
\end{aligned}
\tag{2.14}
$$

in which the two "frequencies" $\alpha_\pm(\mathcal{H})$ are

$$
\alpha_\pm(\mathcal{H}) = \big(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2+4R_0(\mathcal{H})}\,\big)/2, \tag{2.15}
$$

$$
\alpha_+(\mathcal{H})+\alpha_-(\mathcal{H}) = R_1(\mathcal{H}), \quad \alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) = -R_0(\mathcal{H}). \tag{2.16}
$$

If the quantum effects are neglected, *i.e.* $R_1 \equiv 0$ and $\mathcal{H} \to \mathcal{H}_0$ (in the r.h.s.), we have $\alpha_+ = -\alpha_- = \sqrt{R_0(\mathcal{H}_0)}$, the above Heisenberg operator solution reduces to the classical one (2.10) in terms of the quantum-classical correspondence:

$$
[A,B]/i\hbar \to \{A,B\}, \quad (\hbar \to 0). \tag{2.17}
$$

The above exact operator solution looks slightly simpler if the shifted sinusoidal coordinate is used

$$
e^{it\mathcal{H}}\tilde\eta(x)e^{-it\mathcal{H}} = [\mathcal{H},\tilde\eta(x)]\frac{e^{i\alpha_+(\mathcal{H})t}-e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})} + \tilde\eta(x)\frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t}+\alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})}. \tag{2.18}
$$

Like the exact classical solution (2.10), the exact quantum solution (2.14) contains all the dynamical information of the quantum system. One can, for example, determine the entire discrete spectrum $\{\mathcal{E}_n\}$ by following Heisenberg and Pauli's arguments for the harmonic oscillator and the Hydrogen atom. Let us first note that the ground state energy $\mathcal{E}_0 = 0$ is known explicitly, because of our choice of the factorised form of the exactly solvable Hamiltonian (see examples below):

$$
\mathcal{H} = \mathcal{A}^\dagger\mathcal{A}/2, \quad \mathcal{A}\phi_0 = 0 \Rightarrow \mathcal{H}\phi_0 = 0, \quad \mathcal{E}_0 = 0. \tag{2.19}
$$

Let us apply (2.14) to the n-th eigenvector ϕ_n:

$$
\begin{aligned}
&e^{it(\mathcal{H}-\mathcal{E}_n)}\eta(x)\,\phi_n \\
&= \Big([\mathcal{H},\eta(x)]\phi_n + \big(-\eta(x)\alpha_-(\mathcal{E}_n)+R_{-1}(\mathcal{E}_n)/\alpha_+(\mathcal{E}_n)\big)\phi_n\Big)e^{i\alpha_+(\mathcal{E}_n)t}/\big(\alpha_+(\mathcal{E}_n)-\alpha_-(\mathcal{E}_n)\big) \\
&\quad + \Big(-[\mathcal{H},\eta(x)]\phi_n + \big(\eta(x)\alpha_+(\mathcal{E}_n)-R_{-1}(\mathcal{E}_n)/\alpha_-(\mathcal{E}_n)\big)\phi_n\Big)e^{i\alpha_-(\mathcal{E}_n)t}/\big(\alpha_+(\mathcal{E}_n)-\alpha_-(\mathcal{E}_n)\big) \\
&\quad - \big(R_{-1}(\mathcal{E}_n)/R_0(\mathcal{E}_n)\big)\phi_n.
\end{aligned}
\tag{2.20}
$$

Since the r.h.s. has only two different time dependence except for the constant term, the l.h.s. can only have two non-vanishing matrix elements when sandwiched by ϕ_m, except for the obvious ϕ_n corresponding to the constant term. In accordance with the general structure of the eigenfunctions (2.3), they are $\phi_{n\pm1}$:

$$\langle\phi_m|\eta(x)|\phi_n\rangle = 0, \quad \text{for} \quad m \neq n \pm 1, n. \tag{2.21}$$

This imposes the following conditions on the energy eigenvalues

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \alpha_+(\mathcal{E}_n), \quad \mathcal{E}_{n-1} - \mathcal{E}_n = \alpha_-(\mathcal{E}_n). \tag{2.22}$$

Likewise we obtain the 'hermitian conjugate' conditions

$$\mathcal{E}_n - \mathcal{E}_{n-1} = \alpha_+(\mathcal{E}_{n-1}), \quad \mathcal{E}_n - \mathcal{E}_{n+1} = \alpha_-(\mathcal{E}_{n+1}) \tag{2.23}$$

relating these three neighbouring eigenvalues. These overdetermined conditions (2.22), (2.23) and $\mathcal{E}_0 = 0$ determine the entire energy spectrum $\{\mathcal{E}_n\}$ completely for each Hamiltonian. The consistency of the procedure requires that the second term on r.h.s. of (2.20) should vanish when applied to the ground state ϕ_0:

$$-[\mathcal{H}, \eta(x)]\phi_0 + \big(\eta(x)\alpha_+(0) - R_{-1}(0)/\alpha_-(0)\big)\phi_0 = 0, \tag{2.24}$$

which could be interpreted as the equation determining the ground state eigenvector ϕ_0 in the Heisenberg picture. If the number of the discrete levels is finite $(M+1)$ a corresponding condition must be met that the first term on r.h.s. of (2.20) should not belong to the Hilbert space of normalisable vectors when applied to the highest discrete level eigenvector ϕ_M:

$$\big|\big|[\mathcal{H}, \eta(x)]\phi_M + \big(-\eta(x)\alpha_-(\mathcal{E}_M) + R_{-1}(\mathcal{E}_M)/\alpha_+(\mathcal{E}_M)\big)\phi_M\big|\big| = \infty. \tag{2.25}$$

As is clear by now, (2.14) and (2.21) are the physical embodiment of the **three term recursion relations** satisfied by any orthogonal polynomial of single variable:

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta). \tag{2.26}$$

The coefficients A_n, B_n and C_n are also real for a real polynomial $P_n(\eta)$. For the systems treated in this paper, that is those having the general structure of the eigenvectors (2.3), this implies the three term recurrence relations of the eigenfunctions

$$\eta(x)\phi_n(x) = A_n\phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x). \tag{2.27}$$

At the same time the above arguments and the treatment of the harmonic oscillator by Heisenberg clearly show that the operator coefficient of $e^{it\alpha_-(\mathcal{H})}$ on the r.h.s. of the Heisenberg operator solution (2.14) is the annihilation operator, that is, acting on ϕ_n it produces a state ϕ_{n-1}. Likewise, the operator coefficient of $e^{it\alpha_+(\mathcal{H})}$ of the Heisenberg operator solution (2.14) is the creation operator. Thus we arrive at a *dynamical and unified definition* of the **annihilation-creation operators**:

$$
\begin{aligned}
&e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} = a^{(+)}(\mathcal{H},\eta)e^{i\alpha_+(\mathcal{H})t} + a^{(-)}(\mathcal{H},\eta)e^{i\alpha_-(\mathcal{H})t} - R_{-1}(\mathcal{H})/R_0(\mathcal{H}), &(2.28)\\
&a^{(\pm)} = a^{(\pm)}(\mathcal{H},\eta)\\
&\quad \stackrel{\text{def}}{=} \Big(\pm[\mathcal{H},\eta(x)] \mp \big(\eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H})\big)\alpha_\mp(\mathcal{H})\Big) \Big/ \big(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\big). &(2.29)
\end{aligned}
$$

When acting on the eigenvector ϕ_n, they read

$$
a^{(\pm)}\phi_n(x) = \frac{\pm 1}{\mathcal{E}_{n+1} - \mathcal{E}_{n-1}}\Big([\mathcal{H},\eta(x)] + (\mathcal{E}_n - \mathcal{E}_{n\mp 1})\eta(x) + \frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm 1} - \mathcal{E}_n}\Big)\phi_n(x). \tag{2.30}
$$

Before going to the detailed discussion of the annihilation-creation operators for various Hamiltonians in section 2.1 and section 3, let us analyse annihilation-creation operators in a more general context. A minimal requirement for annihilation-creation operators is the following:

(0) : annihilation-creation operators map ϕ_n to ϕ_{n-1} and ϕ_{n+1} (up to an overall constant), respectively.

It should be stressed that there is no a priori principle for fixing the normalisation of the operators. Sometimes it is convenient to introduce the annihilation-creation operators with a different normalisation

$$
a'^{(\pm)} \stackrel{\text{def}}{=} a^{(\pm)}\big(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\big) = \pm[\mathcal{H},\eta(x)] \mp \big(\eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H})\big)\alpha_\mp(\mathcal{H}), \tag{2.31}
$$

which gives

$$
a'^{(\pm)}\phi_n(x) = \pm\Big([\mathcal{H},\eta(x)] + (\mathcal{E}_n - \mathcal{E}_{n\mp 1})\eta(x) + \frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm 1} - \mathcal{E}_n}\Big)\phi_n(x). \tag{2.32}
$$

In the r.h.s. the coefficients of the operator $\eta(x)$ and the identity operator depend on n in general.

The annihilation-creation operators of the **harmonic oscillator** have several remarkable properties:

(i) : Annihilation/creation operator is the positive/negative frequency part of the Heisenberg operator of the 'sinusoidal coordinate'.

(ii) : (annihilation operator)†=(creation operator).

(iii) : $\mathcal{H}=$ const.$\times$ (creation operator)(annihilation operator).

The first property (i) is the principle leading to our unified definition of the annihilation-creation operators as in (2.28)–(2.29). Next we show that they are hermitian conjugate to each other. That is, they satisfy the property (ii), too. By using the three-term recursion relation, we obtain

$$e^{it\mathcal{H}}\eta e^{-it\mathcal{H}}\phi_n = e^{it(\mathcal{E}_{n+1}-\mathcal{E}_n)}A_n\phi_{n+1} + B_n\phi_n + e^{it(\mathcal{E}_{n-1}-\mathcal{E}_n)}C_n\phi_{n-1}. \tag{2.33}$$

Comparing this with (2.28) and (2.29), we arrive at

$$a^{(+)}\phi_n = A_n\phi_{n+1}, \quad a^{(-)}\phi_n = C_n\phi_{n-1}, \quad R_{-1}(\mathcal{E}_n)/R_0(\mathcal{E}_n) = -B_n. \tag{2.34}$$

Here use is made of the facts

$$\begin{aligned} &\alpha_+(\mathcal{E}_n) = \mathcal{E}_{n+1}-\mathcal{E}_n, \quad \alpha_-(\mathcal{E}_n) = \mathcal{E}_{n-1}-\mathcal{E}_n,\\ &R_1(\mathcal{E}_n) = \mathcal{E}_{n+1}+\mathcal{E}_{n-1}-2\mathcal{E}_n, \quad R_0(\mathcal{E}_n) = -(\mathcal{E}_{n+1}-\mathcal{E}_n)(\mathcal{E}_{n-1}-\mathcal{E}_n), \end{aligned} \tag{2.35}$$

and that $\alpha_\pm(\mathcal{H})$ and $R_i(\mathcal{H})$ $(i=1,0,-1)$ are hermitian. Hermitian conjugate of $a^{(-)}$ is

$$a^{(-)\dagger} = \bigl(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\bigr)^{-1}\Bigl([\mathcal{H},\eta(x)] + \alpha_+(\mathcal{H})\bigl(\eta(x)+R_{-1}(\mathcal{H})/R_0(\mathcal{H})\bigr)\Bigr), \tag{2.36}$$

and its action on ϕ_n is

$$\begin{aligned} a^{(-)\dagger}\phi_n &= \bigl(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\bigr)^{-1}\Bigl(\mathcal{H}\eta\phi_n - \eta\mathcal{E}_n\phi_n + \alpha_+(\mathcal{H})\bigl(\eta\phi_n + R_{-1}(\mathcal{E}_n)/R_0(\mathcal{E}_n)\phi_n\bigr)\Bigr)\\ &= \bigl(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\bigr)^{-1}\Bigl((\mathcal{E}_{n+1}-\mathcal{E}_n)A_n\phi_{n+1} + (\mathcal{E}_{n-1}-\mathcal{E}_n)C_n\phi_{n-1}\\ &\qquad\qquad + \alpha_+(\mathcal{H})(A_n\phi_{n+1}+C_n\phi_{n-1})\Bigr)\\ &= \bigl(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\bigr)^{-1}(\mathcal{E}_{n+2}-\mathcal{E}_n)A_n\phi_{n+1}\\ &= A_n\phi_{n+1} = a^{(+)}\phi_n. \end{aligned} \tag{2.37}$$

Therefore $a^{(\pm)}$ are hermitian conjugate to each other, $a^{(-)\dagger} = a^{(+)}$. This also means that

$$e^{-it\alpha_-(\mathcal{H})}a^{(+)} = a^{(+)}e^{it\alpha_+(\mathcal{H})}, \quad e^{-it\alpha_+(\mathcal{H})}a^{(-)} = a^{(-)}e^{it\alpha_-(\mathcal{H})}, \tag{2.38}$$

reflecting the obvious hermiticity of the l.h.s. of (2.28). Note that $a'^{(\pm)}$ are not hermitian conjugate to each other, $a'^{(-)\dagger} \neq a'^{(+)}$, in general.

In the special case of the equi-spaced spectrum $\mathcal{E}_n = an$ (a : constant), to which many interesting examples belong including the harmonic oscillator and its deformation, we have $\alpha_\pm(\mathcal{H}) = \pm a$, $R_1(\mathcal{H}) = 0$, $R_0(\mathcal{H}) = a^2$ and

$$2aa^{(\pm)} = a'^{(\pm)} = \pm[\mathcal{H}, \eta(x)] + a\eta(x) + R_{-1}(\mathcal{H})/a. \tag{2.39}$$

For the simplest harmonic oscillator $\mathcal{H} = (p + ix)(p - ix)/2$, we have $\eta(x) = x$, $R_0 = 1$, $R_1 = R_{-1} = 0$ and $[\mathcal{H}, x] = -ip$ and $2a^{(+)} = x - ip$, $2a^{(-)} = x + ip$, which differ from the conventional ones by a factor $\sqrt{2}$.

In contrast to the above two properties, the third the property (iii) of the annihilation-creation operators is achieved by a very specific modification of the definition as follows:

$$a''^{(-)} \stackrel{\text{def}}{=} a^{(-)}f(\mathcal{H}), \quad a''^{(+)} \stackrel{\text{def}}{=} a''^{(-)\dagger} = f(\mathcal{H})a^{(+)}, \tag{2.40}$$

where $f(\mathcal{H}) = f(\mathcal{H})^\dagger$ is an as yet unspecified function of $\mathcal{H}$. Then we have

$$a''^{(+)}a''^{(-)}\phi_n = f(\mathcal{E}_n)^2 A_{n-1}C_n\phi_n. \tag{2.41}$$

If $f(\mathcal{H})$ is chosen to satisfy

$$f(\mathcal{E}_n)^2 = \mathcal{E}_n/(A_{n-1}C_n) \ \bigl(\stackrel{\text{def}}{=} g(n) \bigr), \tag{2.42}$$

then we obtain

$$\mathcal{H} = a''^{(+)}a''^{(-)}. \tag{2.43}$$

Such an operator $f(\mathcal{H})$ function can be constructed as $f(\mathcal{H}) = \sqrt{g(\mathcal{N})}$, where the number (or level) operator $\mathcal{N}$ is defined by

$$\mathcal{N}\phi_n = n\phi_n. \tag{2.44}$$

This operator $\mathcal{N}$ can be expressed in terms of the Hamiltonian, for example,

$$\mathcal{E}_n = an \quad \Rightarrow \quad \mathcal{N} = \mathcal{H}/a, \tag{2.45}$$

$$\mathcal{E}_n = an^2 + bn \quad \Rightarrow \quad \mathcal{N} = \big(\sqrt{4a\mathcal{H} + b^2} - b\big)/(2a), \tag{2.46}$$

$$\mathcal{E}_n = a(q^{-n} - 1)(1 - bq^n) \quad \Rightarrow \quad q^{\mathcal{N}} = \big(\mathcal{H}/a + b + 1 - \sqrt{(\mathcal{H}/a + b + 1)^2 - 4b}\big)/(2b), \tag{2.47}$$

where a, b, q are constant ($b > 0$ in the second equation, $b < 1$ in the third equation). This $a''^{(\pm)}$ satisfies the property (ii) by definition, but the property (i) becomes ugly or unnatural in general.

Let us close the general theory by a brief discussion of the coherent states. One definition of the coherent state ψ is the eigenvector of the annihilation operator (AOCS, Annihilation Operator Coherent State):

$$a^{(-)}\psi = \lambda\psi, \quad \lambda \in \mathbb{C}. \tag{2.48}$$

In terms of the simple parametrisation $\psi = \sum_{n=0}^{\infty} c_n \phi_n(x)$ (with $c_0 = 1$ as normalization) and the formula (2.34), we arrive at

$$\psi = \psi(\lambda, x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{k=1}^{n} C_n} \cdot P_n(\eta(x)). \tag{2.49}$$

For the equi-spaced spectrum $\mathcal{E}_n = an$ (a : constant), the coherent state has the property of temporal stability

$$e^{it\mathcal{H}}\psi(\lambda, x) = \psi(e^{iat}\lambda, x). \tag{2.50}$$

It should be remarked that the concrete form of the AOCS depends on the specific normalisation of the annihilation operator. For the annihilation operators $a'^{(-)}$, $a''^{(-)}$ and others, we denote the corresponding coherent states as ψ', ψ'', etc. Which coherent state is useful depends on the physics of the system.

2.1 Some typical examples

Now let us look at typical examples [1, 2, 3] to show the actual content of our new unified theory of annihilation-creation operators. In their pioneering work, Nieto and Simmons [4] treated four solvable cases, those discussed in sections 2.1.1, 3.1.1, 3.1.3 and 3.1.4. Some of our results were reported in [4]. Here and throughout this paper we put the dimensionfull quantities as unity, including the Planck's constant.

2.1.1 $1/\sin^2 x$ potential, or symmetric Pöschl-Teller potential

The first example has the $1/\sin^2 x$ potential, which is the one-body case of the well-known Sutherland model [10, 11]. This provides the simplest example of the annihilation-creation operators depending on n. The corresponding coherent state (2.72) or (2.75) had not yet been known, to the best of our knowledge. The system is confined in a finite interval, say $(0, \pi)$ and it has an infinite number of discrete eigenstates. Although this potential is a special ($g = h$) case of the Pöschl-Teller potential discussed in section 3.1.2, it merits separate analysis. The Hamiltonian, the eigenvalues and the eigenfunctions are as follows:

$$\mathcal{H} \stackrel{\text{def}}{=} (p - ig\cot x)(p + ig\cot x)/2, \quad \big(\stackrel{\text{Q.M.}}{\Longrightarrow} 2\mathcal{H} + g^2 = p^2 + g(g-1)/\sin^2 x\big), \tag{2.51}$$

$$\mathcal{E}_n = n(n/2 + g), \quad n = 0, 1, 2, \ldots, \quad g > 0, \quad 0 < x < \pi, \quad \eta(x) = \cos x, \tag{2.52}$$

$$\phi_n(x) = (\sin x)^g P_n^{(\beta,\beta)}(\cos x), \quad \beta \stackrel{\text{def}}{=} g - 1/2, \tag{2.53}$$

in which $P_n^{(\alpha,\beta)}(\eta)$ is the Jacobi polynomial (C.8) and $P_n^{(\beta,\beta)}(\eta)$ is proportional to the Gegenbauer polynomial $C_n^{(\beta+1/2)}(\eta)$ (C.9)

$$\frac{P_n^{(\beta,\beta)}(\eta)}{(\beta+1)_n} = \frac{C_n^{(\beta+1/2)}(\eta)}{(2\beta+1)_n}. \tag{2.54}$$

Hereafter we often use the Pochhammer symbol $(a)_n$, see (C.1).

It is straightforward to evaluate the Poisson brackets

$$\{\mathcal{H}, \cos x\} = p\sin x, \quad \{\mathcal{H}, \{\mathcal{H}, \cos x\}\} = -\cos x\ 2\mathcal{H}', \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} + g^2/2, \tag{2.55}$$

leading to the solution of the initial value problem:

$$\cos x(t) = \cos x(0)\cos\big[t\sqrt{2\mathcal{H}_0'}\,\big] - p(0)\sin x(0)\frac{\sin\big[t\sqrt{2\mathcal{H}_0'}\,\big]}{\sqrt{2\mathcal{H}_0'}}. \tag{2.56}$$

It is straightforward to verify $|\cos x(t)| < 1$. The corresponding quantum expressions are

$$[\mathcal{H}, \cos x] = i\sin x\, p + \cos x/2, \tag{2.57}$$

$$[\mathcal{H}, [\mathcal{H}, \cos x]\,] = \cos x(2\mathcal{H}' - 1/4) + [\mathcal{H}, \cos x], \tag{2.58}$$

$$\alpha_\pm(\mathcal{H}) = 1/2 \pm \sqrt{2\mathcal{H}'}. \tag{2.59}$$

The exact operator solution reads

$$\begin{aligned} e^{it\mathcal{H}}\cos x\, e^{-it\mathcal{H}} &= (i\sin x\, p + \cos x/2)\frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}} \\ &\quad + \cos x\, \frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}}. \end{aligned} \tag{2.60}$$

The annihilation and creation operators are

$$a'^{(\pm)} = a^{(\pm)} 2\sqrt{2\mathcal{H}'} = \pm i \sin x\, p + \cos x \sqrt{2\mathcal{H}'} = \pm \sin x \frac{d}{dx} + \cos x \sqrt{2\mathcal{H}'}. \tag{2.61}$$

It is now obvious that they ($a'^{(\pm)}$) are not hermitian conjugate to each other. The square root sign is neatly removed when applied to the eigenvector ϕ_n as $2\mathcal{E}_n + g^2 = (n+g)^2$:

$$a'^{(-)}\phi_n = -\sin x \frac{d\phi_n}{dx} + (n+g)\cos x\, \phi_n = (n+\beta)\phi_{n-1}, \tag{2.62}$$

$$a'^{(+)}\phi_n = \quad \sin x \frac{d\phi_n}{dx} + (n+g)\cos x\, \phi_n = \frac{2(n+1)(n+2g)}{2n+2g+1}\phi_{n+1}. \tag{2.63}$$

This is a rule rather than exception as expected from the relations between the neighbouring energy levels, (2.22) and (2.23). The right hand sides are the results of the application. In particular, when acting on the ground state ϕ_0, the annihilation ($a'^{(-)}$) and creation ($a'^{(+)}$) operators are proportional to the factorisation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ of the Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}/2$, respectively:

$$\begin{aligned} a'^{(-)}\phi_0 &= \bigl(-\sin x \frac{d}{dx} + g\cos x\bigr)\phi_0 = -\sin x\bigl(\frac{d}{dx} - g\cot x\bigr)\phi_0 = \eta'(x)\mathcal{A}\phi_0 = 0, \\ a'^{(+)}\phi_0 &= \bigl(\sin x \frac{d}{dx} + g\cos x\bigr)\phi_0 = -\sin x\bigl(-\frac{d}{dx} - g\cot x\bigr)\phi_0 = \eta'(x)\mathcal{A}^\dagger\phi_0. \end{aligned} \tag{2.64}$$

In a rough sense, the factor $\eta'(x) = -\sin x$, in the creation operator, compensates the downward shift of the parameter (g) caused by $\mathcal{A}^\dagger$. Similar situations are encountered in all the other quantum systems. In particular, for the systems with equi-spaced spectrum $\mathcal{E}_n = an$ (a : constant), the factorisation of $a'^{(-)}$ and $a'^{(+)}$ into $\mathcal{A}$ and $\mathcal{A}^\dagger$ is n-independent, (2.94), (3.14), (3.15), (3.92). Their significance will be discussed in some detail for the 'discrete' quantum mechanics cases in Appendix **B**.

The following interesting commutation relations ensue from (2.62) and (2.63):

$$[\mathcal{H}, a'^{(\pm)}] = \pm\bigl(\sqrt{2\mathcal{H}'}\, a'^{(\pm)} + a'^{(\pm)}\sqrt{2\mathcal{H}'}\,\bigr)/2, \tag{2.65}$$

$$[a'^{(-)}, a'^{(+)}] = 2\sqrt{2\mathcal{H}'}, \tag{2.66}$$

$$a'^{(-)}a'^{(+)} + a'^{(+)}a'^{(-)} = 4\mathcal{H} + 2g. \tag{2.67}$$

The relation (2.67) could be accepted as a substitute of the property (iii) of the annihilation-creation operators discussed in page 9.

By the similarity transformation in terms of the ground state wavefunction $\phi_0(x) = (\sin x)^g$, we obtain the so-called shift down and up operators for the Jacobi (Gegenbauer)

polynomial ($\beta = g - 1/2$):

$$\text{down} \ : \quad (1-\eta^2)\frac{d}{d\eta}P_n^{(\beta,\beta)}(\eta) + n\eta P_n^{(\beta,\beta)}(\eta) = (n+\beta)P_{n-1}^{(\beta,\beta)}(\eta), \tag{2.68}$$

$$\text{up} \ : \ -(1-\eta^2)\frac{d}{d\eta}P_n^{(\beta,\beta)}(\eta) + (n+2g)\eta P_n^{(\beta,\beta)}(\eta) = \frac{2(n+1)(n+2g)}{2n+2g+1}P_{n+1}^{(\beta,\beta)}(\eta). \tag{2.69}$$

As expected they are the same Jacobi polynomials of degree $n-1$ and $n+1$. It should be stressed that these shift down-up operators are naturally derived from our annihilation-creation operators without assuming the explicit form of the three term recursion relation.

The coherent state ψ (2.48) is

$$\psi(x) = \phi_0(x)\sum_{n=0}^{\infty}(2\lambda)^n\frac{(\beta+3/2)_n}{(\beta+1)_n}P_n^{(\beta,\beta)}(\cos x) = \phi_0(x)\sum_{n=0}^{\infty}(2\lambda)^n\frac{(g+1)_n}{(2g)_n}C_n^{(g)}(\cos x), \tag{2.70}$$

where we have used $a^{(-)}\phi_n = a'^{(-)}\phi_n/(2(n+g))$, (2.62) and (2.54). A generating function of the Gegenbauer polynomials (γ: arbitrary) [6],

$$\sum_{n=0}^{\infty}t^n\frac{(\gamma)_n}{(2g)_n}C_n^{(g)}(\eta) = (1-\eta t)^{-\gamma}\,{}_2F_1\Bigl(\begin{matrix}\gamma/2, (\gamma+1)/2\\ g+1/2\end{matrix}\ \Big|\ \frac{(\eta^2-1)t}{(1-\eta t)^2}\Bigr), \tag{2.71}$$

gives a concise expression of the coherent state ψ:

$$\psi(x) = (1-2\lambda\cos x)^{-g-1}\,{}_2F_1\Bigl(\begin{matrix}(g+1)/2, g/2+1\\ g+1/2\end{matrix}\ \Big|\ \frac{-2\lambda\sin^2 x}{(1-2\lambda\cos x)^2}\Bigr). \tag{2.72}$$

Here ${}_2F_1$ is the hypergeometric function (C.3). For the annihilation operator $a'^{(-)}$, the corresponding coherent state ψ' is

$$\psi'(x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{\lambda^n}{(\beta+1)_n}P_n^{(\beta,\beta)}(\cos x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{\lambda^n}{(2g)_n}C_n^{(g)}(\cos x). \tag{2.73}$$

A generating function of the Gegenbauer polynomials [6],

$$\sum_{n=0}^{\infty}\frac{t^n}{(2g)_n}C_n^{(g)}(\eta) = e^{\eta t}{}_0F_1\Bigl(\begin{matrix}-\\ g+1/2\end{matrix}\ \Big|\ \frac{(\eta^2-1)t^2}{4}\Bigr), \tag{2.74}$$

gives a concise expression of the coherent state ψ':

$$\psi'(x) = \Gamma(g+1/2)\,e^{\lambda\cos x}(\lambda/2)^{1/2-g}\sqrt{\sin x}\,J_{g-1/2}(\lambda\sin x), \tag{2.75}$$

in which $J_a(x)$ is the Bessel function (C.5).

2.1.2 Deformed harmonic oscillator ⇒ Meixner-Pollaczek polynomial

The deformed harmonic oscillator is a simplest example of shape invariant 'discrete' quantum mechanics. The Hamiltonian of 'discrete' quantum mechanics studied in this paper has the following form [2] (with some modification for the Askey-Wilson case in section 2.1.3):

$$\mathcal{H} \stackrel{\text{def}}{=} \Big(\sqrt{V(x)}\, e^{p}\sqrt{V(x)^*} + \sqrt{V(x)^*}\, e^{-p}\sqrt{V(x)} - V(x) - V(x)^*\Big)\,/\,2. \tag{2.76}$$

The eigenvalue problem for $\mathcal{H}$, $\mathcal{H}\phi = \mathcal{E}\phi$ is a difference equation, instead of a second order differential equation. Let us define $S_\pm$, $T_\pm$ and $\mathcal{A}$ by

$$S_+ \stackrel{\text{def}}{=} e^{p/2}\sqrt{V(x)^*},\ \ S_- \stackrel{\text{def}}{=} e^{-p/2}\sqrt{V(x)},\ \ S_+^\dagger = \sqrt{V(x)}\, e^{p/2},\ \ S_-^\dagger = \sqrt{V(x)^*}\, e^{-p/2}, \tag{2.77}$$

$$T_+ \stackrel{\text{def}}{=} S_+^\dagger S_+ = \sqrt{V(x)}\, e^{p}\sqrt{V(x)^*},\quad T_- \stackrel{\text{def}}{=} S_-^\dagger S_- = \sqrt{V(x)^*}\, e^{-p}\sqrt{V(x)}, \tag{2.78}$$

$$\mathcal{A} \stackrel{\text{def}}{=} i(S_+ - S_-),\quad \mathcal{A}^\dagger = -i(S_+^\dagger - S_-^\dagger). \tag{2.79}$$

Then the Hamiltonian is factorized

$$\mathcal{H} = \big(T_+ + T_- - V(x) - V(x)^*\big)/2 = (S_+^\dagger - S_-^\dagger)(S_+ - S_-)/2 = \mathcal{A}^\dagger\mathcal{A}/2. \tag{2.80}$$

The potential function $V(x)$ of the deformed harmonic oscillator is

$$V(x) = a + ix,\quad -\infty < x < \infty,\quad a > 0. \tag{2.81}$$

As shown in some detail in our previous paper [2], it has an equi-spaced spectrum and the corresponding eigenfunctions are a special case of the Meixner-Pollaczek polynomial $P_n^{(a)}(x\,;\frac{\pi}{2})$ (C.10),

$$\mathcal{E}_n = n,\quad n = 0, 1, 2, \ldots, \tag{2.82}$$

$$\phi_0(x) = \sqrt{\Gamma(a+ix)\Gamma(a-ix)},\quad \eta(x) = x, \tag{2.83}$$

$$\phi_n(x) = \phi_0(x)P_n(x),\quad P_n(x) \stackrel{\text{def}}{=} P_n^{(a)}(x\,;\tfrac{\pi}{2}), \tag{2.84}$$

which could be considered as a deformation of the Hermite polynomial.

The Poisson bracket relations are

$$\{\mathcal{H}, x\} = -\sqrt{a^2+x^2}\sinh p,\qquad \{\mathcal{H},\{\mathcal{H}, x\}\} = -x, \tag{2.85}$$

leading to the harmonic oscillation,

$$x(t) = x(0)\cos t + \sqrt{a^2 + x^2(0)}\,\sinh p(0)\,\sin t, \tag{2.86}$$

which endorses the naming of the deformed harmonic oscillator. The corresponding quantum expressions are also simple:

$$[\mathcal{H}, x] = -i(T_+ - T_-)/2, \qquad [\mathcal{H}, [\mathcal{H}, x]\,] = x, \tag{2.87}$$

$$e^{it\mathcal{H}}\, x\, e^{-it\mathcal{H}} = x\, \cos t + i[\mathcal{H}, x]\, \sin t = x\, \cos t + (T_+ - T_-)/2 \sin t. \tag{2.88}$$

The annihilation and creation operators are

$$a'^{(\pm)} = 2a^{(\pm)} = x \pm [\mathcal{H}, x] = x \mp i(T_+ - T_-)/2, \tag{2.89}$$

which are hermitian conjugate to each other. These operators were also introduced by Degasperis and Ruijsenaars [12] by a different reasoning from ours. By similarity transformation in terms of the ground state wavefunction $\phi_0(x) = \sqrt{\Gamma(a + ix)\Gamma(a - ix)}$, we obtain

$$\phi_0(x)^{-1}\ a'^{(\pm)}\ \phi_0(x) = x \mp i\big(V(x)e^{p} - V(x)^{*}e^{-p}\big)/2\,. \tag{2.90}$$

The action of the annihilation creation operators on the eigenvectors

$$a'^{(-)}\phi_n = (n + 2a - 1)\phi_{n-1}, \quad a'^{(+)}\phi_n = (n + 1)\phi_{n+1} \tag{2.91}$$

is consistent with the three term recurrence relation of the Meixner-Pollaczek polynomial:

$$(n + 1)P^{(a)}_{n+1}(x\,; \tfrac{\pi}{2}) - 2xP^{(a)}_{n}(x\,; \tfrac{\pi}{2}) + (n + 2a - 1)P^{(a)}_{n-1}(x\,; \tfrac{\pi}{2}) = 0. \tag{2.92}$$

From these it is easy to verify the $\mathfrak{su}(1,1)$ commutation relations including the Hamiltonian $\mathcal{H}$:

$$[\mathcal{H}, a'^{(\pm)}] = \pm\, a'^{(\pm)}, \quad [a'^{(-)}, a'^{(+)}] = 2(\mathcal{H} + a). \tag{2.93}$$

It is interesting to note that $a^{(\pm)}$ are factorised by the factors of the Hamiltonian $\mathcal{H} = \mathcal{A}^{\dagger}\mathcal{A}/2$,

$$4a^{(-)} = X^{\dagger}\mathcal{A}, \quad 4a^{(+)} = \mathcal{A}^{\dagger}X, \tag{2.94}$$

$$X \stackrel{\text{def}}{=} S_+ + S_-, \quad X^{\dagger} = S_+^{\dagger} + S_-^{\dagger}. \tag{2.95}$$

These X and $X^{\dagger}$ compensate the shift of the parameter a caused by $\mathcal{A}^{\dagger}$ and $\mathcal{A}$. See Appendix **B** for more details.

The coherent state (2.48), (2.49), is simply obtained from the formula (2.91) and $a'^{(-)} = 2a^{(-)}$:

$$\psi(x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{(2\lambda)^n}{(2a)_n} P^{(a)}_{n}(x\,; \tfrac{\pi}{2}). \tag{2.96}$$

A generating function of the Meixner-Pollaczek polynomial [6],

$$\sum_{n=0}^{\infty}\frac{t^n}{(2a)_n}P_n^{(a)}(x\,;\tfrac{\pi}{2})=e^{it}{}_1F_1\Bigl(\begin{matrix}a+ix\\ 2a\end{matrix}\Bigm|-2it\Bigr), \tag{2.97}$$

gives a concise expression of the coherent state ψ:

$$\psi(x)=\phi_0(x)\,e^{2i\lambda}\,{}_1F_1\Bigl(\begin{matrix}a+ix\\ 2a\end{matrix}\Bigm|-4i\lambda\Bigr), \tag{2.98}$$

in which ${}_1F_1$ is the hypergeometric function (C.3).

2.1.3 Askey-Wilson polynomial

The Askey-Wilson polynomial belongs to the so-called q-scheme of hypergeometric polynomials [6]. It has four parameters a_1,a_2,a_3,a_4 on top of q $(0<q<1)$, and is considered as a three-parameter deformation of the Jacobi polynomial. As shown in our previous papers [3, 2], it also describes the equilibrium positions of the trigonometric Ruijsenaars-Schneider systems based on the BC root system [13]. Thus, as a dynamical system, it could be called a deformed Pöschl-Teller potential or one body case of the trigonometric BC Ruijsenaars-Schneider systems. The quantum-classical correspondence has some more subtlety than the other ‘discrete’ quantum mechanical systems treated in section 3.2 because of another ‘classical’ limit $q\to 1$.

The factorised Hamiltonian of the Askey-Wilson polynomial has a bit different form from that of the Meixner-Pollaczek polynomial (2.76):

$$\mathcal{H}\stackrel{\text{def}}{=}\Bigl(\sqrt{V(z)}\,q^D\sqrt{V(z)^*}+\sqrt{V(z)^*}\,q^{-D}\sqrt{V(z)}-V(z)-V(z)^*\Bigr)\big/\,2, \tag{2.99}$$

with a potential function $V(z)$:

$$V(z)=\frac{\prod_{j=1}^4(1-a_jz)}{(1-z^2)(1-qz^2)},\quad z=e^{ix},\quad 0<x<\pi,\quad D\stackrel{\text{def}}{=}z\frac{d}{dz}=-i\frac{d}{dx}=p. \tag{2.100}$$

We assume $-1<a_1,a_2,a_3,a_4<1$ and $a_1a_2a_3a_4<q$. This Hamiltonian is also factorised $\mathcal{H}=\mathcal{A}^\dagger\mathcal{A}/2$, where $\mathcal{A}$ and $\mathcal{A}^\dagger$ are given in (2.77)–(2.79) with the replacement $V(x)\Rightarrow V(z)$, $e^{\pm p/2}\Rightarrow q^{\pm D/2}$, etc. The eigenvalues and eigenfunctions are [2, 3]:

$$\mathcal{E}_n=(q^{-n}-1)(1-a_1a_2a_3a_4q^{n-1})/2,\quad n=0,1,2,\ldots, \tag{2.101}$$

$$\phi_0(x)=\sqrt{\frac{(z^2\,;q)_\infty}{\prod_{j=1}^4(a_jz\,;q)_\infty}\frac{(z^{-2}\,;q)_\infty}{\prod_{j=1}^4(a_jz^{-1}\,;q)_\infty}},\quad \eta(x)=\frac{z+z^{-1}}{2}=\cos x, \tag{2.102}$$

$$\phi_n(x)=\phi_0(x)P_n(\cos x),\quad P_n(\eta)\stackrel{\text{def}}{=}p_n(\eta\,;a_1,a_2,a_3,a_4|q), \tag{2.103}$$

in which $p_n(\eta\,;a_1,a_2,a_3,a_4|q)$ is the Askey-Wilson polynomial (C.14).

The presence of the q-factor has only superficial effects at the classical level with the Hamiltonian ($\gamma=\log q$):

$$\mathcal{H}_c=\sqrt{V_c(z)V_c(z)^*}\cosh\gamma p-\bigl(V_c(z)+V_c(z)^*\bigr)/2,\quad V_c(z)=\frac{\prod_{j=1}^4(1-a_jz)}{(1-z^2)^2}, \tag{2.104}$$

$$\{\mathcal{H}_c,\cos x\}=\gamma\sqrt{\prod_{j=1}^4(1-a_jz)\prod_{j=1}^4(1-a_j/z)}\,/\,(4\sin x)\ \sinh\gamma p, \tag{2.105}$$

$$\{\mathcal{H}_c,\{\mathcal{H}_c,\cos x\}\}=-\cos x\,R_0(\mathcal{H}_c)-R_{-1}(\mathcal{H}_c), \tag{2.106}$$

$$R_0(\mathcal{H}_c)=\gamma^2(\mathcal{H}_c^2+c_1\mathcal{H}_c+c_2),\quad R_{-1}(\mathcal{H}_c)=-\gamma^2(c_3\mathcal{H}_c+c_4), \tag{2.107}$$

with coefficients c_1,..,c_4:

$$c_1=1+b_4,\quad c_2=(1-b_4)^2/4,\quad c_3=(b_1+b_3)/4,\quad c_4=(1-b_4)(b_1-b_3)/8. \tag{2.108}$$

Here we use the abbreviation

$$b_1\stackrel{\text{def}}{=}\sum_{1\le j\le 4}a_j\,,\quad b_3\stackrel{\text{def}}{=}\sum_{1\le j<k<l\le 4}a_ja_ka_l\,,\quad b_4\stackrel{\text{def}}{=}\prod_{j=1}^4 a_j\,. \tag{2.109}$$

The corresponding quantum expressions are

$$[\mathcal{H},\cos x]=(q^{-1}-1)\bigl(z^{-1}(1-qz^2)T_++z(1-qz^{-2})T_-\bigr)/4, \tag{2.110}$$

$$[\mathcal{H},[\mathcal{H},\cos x]\,]=\cos x\,R_0(\mathcal{H})+[\mathcal{H},\cos x]R_1(\mathcal{H})+R_{-1}(\mathcal{H}), \tag{2.111}$$

$$R_0(\mathcal{H})=q(q^{-1}-1)^2\Bigl((\mathcal{H}')^2-(1+q^{-1})^2b_4/4\Bigr)\,, \tag{2.112}$$

$$R_1(\mathcal{H})=q(q^{-1}-1)^2\,\mathcal{H}',\qquad \mathcal{H}'\stackrel{\text{def}}{=}\mathcal{H}+(1+q^{-1}b_4)/2, \tag{2.113}$$

$$R_{-1}(\mathcal{H})=-q(q^{-1}-1)^2\Bigl((b_1+q^{-1}b_3)\mathcal{H}/4+(1-q^{-2}b_4)(b_1-b_3)/8\Bigr). \tag{2.114}$$

The two frequencies are:

$$\alpha_\pm(\mathcal{H})=(q^{-1}-1)\Bigl((1-q)\mathcal{H}'\pm(1+q)\sqrt{(\mathcal{H}')^2-q^{-1}b_4}\Bigr)\,/\,2, \tag{2.115}$$

in which

$$\mathcal{H}'\phi_n=(q^{-n}+b_4q^{n-1})/2\ \phi_n,\quad \bigl((\mathcal{H}')^2-q^{-1}b_4\bigr)\phi_n=(q^{-n}-b_4q^{n-1})^2/4\ \phi_n. \tag{2.116}$$

The annihilation-creation operators are:

$$\begin{aligned}a^{(\pm)}=\Bigl(&\pm(q^{-1}-1)\bigl(z^{-1}(1-qz^2)T_++z(1-qz^{-2})T_-\bigr)/4\\&\mp\cos x\,\alpha_\mp(\mathcal{H})\pm R_{-1}(\mathcal{H})\alpha_\pm(\mathcal{H})^{-1}\Bigr)\,/\,\bigl(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\bigr)\,.\end{aligned} \tag{2.117}$$

Their effects on the eigenvectors are:

$$a^{(-)}\phi_n = \frac{(1-q^n)\prod_{1\le j<k\le 4}(1-a_ja_kq^{n-1})}{2(1-b_4q^{2n-2})(1-b_4q^{2n-1})}\,\phi_{n-1}, \tag{2.118}$$

$$a^{(+)}\phi_n = \frac{1-b_4q^{n-1}}{2(1-b_4q^{2n-1})(1-b_4q^{2n})}\,\phi_{n+1}, \tag{2.119}$$

which are consistent with the three term recurrence relation of the Askey-Wilson polynomial. The 'annihilation-creation' operators on the polynomial $P_n(\cos x)$ read

$$\begin{aligned}&\phi_0(x)^{-1}a^{(\pm)}\phi_0(x)\cdot P_n(\cos x)\\ =&\frac{1}{\mathcal{E}_{n+1}-\mathcal{E}_{n-1}}\Big(\pm(q^{-1}-1)\Big(z^{-1}(1-qz^2)V(z)\,q^D+z(1-qz^{-2})V(z)^*q^{-D}\Big)\,/\,4\\ &\qquad\pm(\mathcal{E}_n-\mathcal{E}_{n\mp1})\cos x\pm\frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm1}-\mathcal{E}_n}\Big)P_n(\cos x).\end{aligned} \tag{2.120}$$

The coherent state is

$$\psi(x)=\phi_0(x)\sum_{n=0}^{\infty}\frac{(2\lambda)^n}{(q\,;q)_n}\frac{(a_1a_2a_3a_4\,;q)_{2n}}{\prod_{1\le j<k\le 4}(a_ja_k\,;q)_n}\,P_n(\cos x)\,. \tag{2.121}$$

We are not aware if a concise summation formula exists or not.

3 Various results

In this section we will briefly present many interesting results on the annihilation-creation operators, their algebraic properties and coherent states, etc, for various exactly solvable quantum mechanical systems, including the 'discrete' quantum mechanical systems. All these solvable systems share *shape invariance* [8, 1, 2] which is a purely quantum mechanical notion that guarantees quantum solvability. But that property plays no active role in the present theory. On the other hand, the 'sinusoidal motion' exists at the classical and quantum levels. In Appendix **A** we will show that a system realising the exact sinusoidal motion is quite limited and that all of them belong to the known shape invariant systems. In other words there are shape invariant systems that do not have sinusoidal motion.

3.1 Ordinary quantum mechanical systems

3.1.1 x^2+1/x^2 potential

When a centrifugal barrier (a $1/x^2$ potential) is added, the harmonic oscillator keeps its exact solvability, but the particle is restricted to a half line, either $x>0$ or $x<0$. This is the one-body case of the well-known Calogero model [10, 11]. The eigenfunctions are described by

the Laguerre polynomial and the annihilation-creation operators within the $\mathfrak{su}(1,1)$ scheme are well-known [4, 14]. Our unified theory predicts these operators naturally. The coherent and squeezed states in the $\mathfrak{su}(1,1)$ were already known [14]. Its Hamiltonian, the eigenvalues and the eigenfunctions are:

$$\mathcal{H} \stackrel{\text{def}}{=} (p + ix - ig/x)(p - ix + ig/x)/2, \quad 0 < x < \infty, \quad g > 0, \tag{3.1}$$

$$\mathcal{E}_n = 2n, \quad n = 0, 1, 2, \ldots, \quad \eta(x) = x^2, \tag{3.2}$$

$$\phi_n(x) = \phi_n(x\,; g) = e^{-x^2/2} x^g L_n^{(\beta)}(x^2), \quad \beta \stackrel{\text{def}}{=} g - 1/2, \tag{3.3}$$

in which $L_n^{(\beta)}(\eta)$ is the Laguerre polynomial (C.7).

The Poisson bracket relations are simple

$$\{\mathcal{H}, x^2\} = -2px, \quad \{\mathcal{H}, \{\mathcal{H}, x^2\}\} = -4(x^2 - \mathcal{H} - g), \tag{3.4}$$

leading to the simple sinusoidal motion

$$x^2(t) = x^2(0)\,\cos 2t + (1 - \cos 2t)(\mathcal{H}_0 + g) + p(0)x(0)\sin 2t. \tag{3.5}$$

It is straightforward to verify $x^2(t) > 0$. The quantum theory is almost the same as the classical one:

$$[\mathcal{H}, x^2] = -i(xp + px), \quad [\mathcal{H}, [\mathcal{H}, x^2]\,] = 4(x^2 - \mathcal{H}'), \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} + g + 1/2, \tag{3.6}$$

$$e^{it\mathcal{H}} x^2\, e^{-it\mathcal{H}} = x^2\,\cos 2t + (1 - \cos 2t)\mathcal{H}' + (xp + px)/2\,\sin 2t, \tag{3.7}$$

which leads to the following annihilation and creation operators:

$$a^{(\pm)} = (x^2 - \mathcal{H}')/2 \mp i(xp + px)/4 = \Big(\big(\frac{d}{dx} \mp x\big)^2 - \frac{g(g-1)}{x^2}\Big)\,/\,4. \tag{3.8}$$

The action of these operators are ($\beta = g - 1/2$)

$$a^{(-)}\phi_n = -(n + \beta)\phi_{n-1}, \quad a^{(+)}\phi_n = -(n + 1)\phi_{n+1}, \tag{3.9}$$

which are consistent with the three term recurrence relation of the Laguerre polynomial

$$(n + 1)L_{n+1}^{(\beta)}(\eta) + (\eta - 2n - \beta - 1)L_n^{(\beta)}(\eta) + (n + \beta)L_{n-1}^{(\beta)}(\eta) = 0. \tag{3.10}$$

From these follow the $\mathfrak{su}(1,1)$ relations

$$[\mathcal{H}, a^{(\pm)}] = \pm 2a^{(\pm)}, \quad [a^{(-)}, a^{(+)}] = \mathcal{H}' = \mathcal{H} + g + 1/2. \tag{3.11}$$

The coherent state (AOCS) (2.49) is obtained simply as

$$\psi(x)=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{(\beta+1)_n}\,\phi_n(x)=\phi_0(x)\frac{e^{-\lambda}\Gamma(\beta+1)}{(-x^2\lambda)^{\beta/2}}J_\beta(2x\sqrt{-\lambda}), \tag{3.12}$$

in which a generating function of the Laguerre polynomial [6]

$$\sum_{n=0}^{\infty}\frac{t^n}{(\alpha+1)_n}L_n^{(\alpha)}(x)=e^t\,{}_0F_1\Bigl(\genfrac{}{}{0pt}{}{-}{\alpha+1}\Bigm|-xt\Bigr) \tag{3.13}$$

and (C.5) are used.

The annihilation and creation operators (3.8) are factorised by the factors of the Hamiltonian $\mathcal{H}=\mathcal{A}^\dagger\mathcal{A}/2$:

$$4a^{(-)}=\Bigl(\frac{d}{dx}+x+\frac{g}{x}\Bigr)\cdot\mathcal{A},\qquad \mathcal{A}=i(p-ix+ig/x), \tag{3.14}$$

$$4a^{(+)}=\mathcal{A}^\dagger\cdot\Bigl(-\frac{d}{dx}+x+\frac{g}{x}\Bigr),\quad \mathcal{A}^\dagger=-i(p+ix-ig/x). \tag{3.15}$$

This is a degenerate case of (3.92) and the action of the other factors is to compensate the parameter shifts caused by the operators $\mathcal{A}^\dagger$ and $\mathcal{A}$:

$$\Bigl(-\frac{d}{dx}+x+\frac{g}{x}\Bigr)\phi_n(x\,;g)=2\,\phi_n(x\,;g+1), \tag{3.16}$$

$$\Bigl(\frac{d}{dx}+x+\frac{g}{x}\Bigr)\phi_n(x\,;g+1)=(2n+2g+1)\,\phi_n(x\,;g). \tag{3.17}$$

3.1.2 Pöschl-Teller potential

The Pöschl-Teller potential has two parameters g and h and its eigenfunctions are related to the Jacobi polynomial. It is the one body case of the BC type Sutherland systems [10, 11]. Its Hamiltonian, the eigenvalues and the eigenfunctions are:

$$\mathcal{H}\stackrel{\text{def}}{=}(p-ig\cot x+ih\tan x)(p+ig\cot x-ih\tan x)/2,\quad 0<x<\pi/2, \tag{3.18}$$

$$\mathcal{E}_n=2n(n+g+h),\quad n=0,1,2,\ldots,\quad g,h>0,\quad \eta(x)=\cos 2x, \tag{3.19}$$

$$\phi_n(x)=(\sin x)^g(\cos x)^hP_n^{(\alpha,\beta)}(\cos 2x),\quad \alpha\stackrel{\text{def}}{=}g-1/2,\quad \beta\stackrel{\text{def}}{=}h-1/2, \tag{3.20}$$

in which $P_n^{(\alpha,\beta)}(\eta)$ is the Jacobi polynomial (C.8). It is straightforward to evaluate the Poisson brackets

$$\{\mathcal{H},\cos 2x\}=2p\sin 2x, \tag{3.21}$$

$$\{\mathcal{H},\{\mathcal{H},\cos 2x\}\}=-\cos 2x\,8\mathcal{H}'-4(g^2-h^2),\quad \mathcal{H}'\stackrel{\text{def}}{=}\mathcal{H}+(g+h)^2/2, \tag{3.22}$$

leading to the solution of the initial value problem:

$$\cos 2x(t) = \Big(\cos 2x(0) + \frac{g^2-h^2}{2\mathcal{H}_0'}\Big)\cos\big[2t\sqrt{2\mathcal{H}_0'}\,\big] \\ - p(0)\sin 2x(0)\frac{\sin\big[2t\sqrt{2\mathcal{H}_0'}\,\big]}{\sqrt{2\mathcal{H}_0'}} - \frac{g^2-h^2}{2\mathcal{H}_0'}\,. \tag{3.23}$$

Note that $|\cos 2x(t)| < 1$ is satisfied. The corresponding quantum expressions are

$$[\mathcal{H}, \cos 2x] = 2(i\sin 2x\ p + \cos 2x), \tag{3.24}$$

$$[\mathcal{H}, [\mathcal{H}, \cos 2x]\,] = \cos 2x\,(8\mathcal{H}' - 4) + 4[\mathcal{H}, \cos 2x] + 4(\alpha^2 - \beta^2), \tag{3.25}$$

$$\alpha_\pm(\mathcal{H}) = 2 \pm 2\sqrt{2\mathcal{H}'}. \tag{3.26}$$

The exact operator solution reads

$$e^{it\mathcal{H}}\cos 2x\, e^{-it\mathcal{H}} = (i\sin 2x\ p + \cos 2x)\frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}} - \frac{\alpha^2-\beta^2}{2\mathcal{H}'-1} \\ + \big(\cos 2x\,(2\mathcal{H}'-1) + \alpha^2 - \beta^2\big)\frac{1}{\sqrt{2\mathcal{H}'}}\Big(\frac{e^{i\alpha_+(\mathcal{H})t}}{\alpha_+(\mathcal{H})} - \frac{e^{i\alpha_-(\mathcal{H})t}}{\alpha_-(\mathcal{H})}\Big). \tag{3.27}$$

The annihilation and creation operators are

$$a'^{(\pm)}/2 = a^{(\pm)}2\sqrt{2\mathcal{H}'} = \pm\sin 2x\frac{d}{dx} + \cos 2x\,\sqrt{2\mathcal{H}'} + \frac{\alpha^2-\beta^2}{\sqrt{2\mathcal{H}'}\pm 1}\,. \tag{3.28}$$

It is now obvious that they ($a'^{(\pm)}$) are not hermitian conjugate to each other. When applied to the eigenvector ϕ_n as $2\mathcal{E}_n + (g+h)^2 = (2n+g+h)^2$, we obtain:

$$a'^{(-)}/2\,\phi_n = -\sin 2x\frac{d\phi_n}{dx} + (2n+g+h)\cos 2x\,\phi_n + \frac{\alpha^2-\beta^2}{2n+\alpha+\beta}\,\phi_n \\ = \frac{4(n+\alpha)(n+\beta)}{2n+\alpha+\beta}\,\phi_{n-1}, \tag{3.29}$$

$$a'^{(+)}/2\,\phi_n = \quad \sin 2x\frac{d\phi_n}{dx} + (2n+g+h)\cos 2x\,\phi_n + \frac{\alpha^2-\beta^2}{2n+\alpha+\beta+2}\,\phi_n \\ = \frac{4(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+2}\,\phi_{n+1}. \tag{3.30}$$

The right hand sides are the results of the application. From (3.29) and $a'^{(-)}\phi_n = 4(2n + g + h)a^{(-)}\phi_n$, the coherent state ψ and ψ' are

$$\psi(x) = \phi_0(x)\sum_{n=0}^{\infty}(\lambda/2)^n\frac{(\alpha+\beta+2)_{2n}}{(\alpha+1)_n(\beta+1)_n}\,P_n^{(\alpha,\beta)}(\cos 2x), \tag{3.31}$$

$$\psi'(x) = \phi_0(x)\sum_{n=0}^{\infty}(\lambda/4)^n\frac{(\frac{\alpha+\beta}{2}+1)_n}{(\alpha+1)_n(\beta+1)_n}\,P_n^{(\alpha,\beta)}(\cos 2x). \tag{3.32}$$

We are not aware if concise summation formulas exist or not.

3.1.3 Soliton potential, or the symmetric Rosen-Morse potential

As is well-known $-g(g+1)/\cosh^2 x$ potential is *reflectionless* for integer coupling constant g, corresponding to the KdV soliton. It has a finite number $1+[g]'$ (the greatest integer not equal or exceeding g) of bound states:

$$\mathcal{H} \stackrel{\text{def}}{=} (p+ig\tanh x)(p-ig\tanh x)/2, \quad -\infty < x < \infty, \quad g>0, \tag{3.33}$$

$$\mathcal{E}_n = n(-n/2+g), \quad n=0,1,\ldots,[g]', \quad \eta(x)=\sinh x, \tag{3.34}$$

$$\phi_n(x) = i^{-n}(\cosh x)^{-g} P_n^{(\beta,\beta)}(i\sinh x), \quad \beta \stackrel{\text{def}}{=} -g-1/2. \tag{3.35}$$

These eigenfunctions are real due to the parity $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$. The Poisson brackets are

$$\{\mathcal{H},\sinh x\} = -p\cosh x, \quad \{\mathcal{H},\{\mathcal{H},\sinh x\}\} = \sinh x\ 2\mathcal{H}', \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H}-g^2/2, \tag{3.36}$$

leading to the solution of the initial value problem:

$$\sinh x(t) = \sinh x(0)\cos\left[t\sqrt{-2\mathcal{H}'_0}\,\right] + p(0)\cosh x(0)\frac{\sin\left[t\sqrt{-2\mathcal{H}'_0}\,\right]}{\sqrt{-2\mathcal{H}'_0}}. \tag{3.37}$$

It describes sinusoidal motion for bound states $\mathcal{H}'_0 < 0$ only. But the above expression is valid for the unbound motion $\mathcal{H}'_0 > 0$, too.

The corresponding quantum expressions are

$$[\mathcal{H},\sinh x] = -i\cosh x\ p - \sinh x/2, \tag{3.38}$$

$$[\mathcal{H},[\mathcal{H},\sinh x]\,] = -\sinh x(2\mathcal{H}'+1/4) - [\mathcal{H},\sinh x], \tag{3.39}$$

$$\alpha_\pm(\mathcal{H}) = -1/2 \pm \sqrt{-2\mathcal{H}'}. \tag{3.40}$$

The exact operator solution reads

$$\begin{aligned} e^{it\mathcal{H}}\sinh x\, e^{-it\mathcal{H}} &= (-i\cosh x\ p - \sinh x/2)\,\frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{-2\mathcal{H}'}} \\ &\quad - \sinh x\,\frac{2\mathcal{H}'+1/4}{2\sqrt{-2\mathcal{H}'}}\left(\frac{e^{i\alpha_+(\mathcal{H})t}}{\alpha_+(\mathcal{H})} - \frac{e^{i\alpha_-(\mathcal{H})t}}{\alpha_-(\mathcal{H})}\right). \end{aligned} \tag{3.41}$$

The annihilation and creation operators are

$$a'^{(\pm)} = a^{(\pm)} 2\sqrt{-2\mathcal{H}'} = \mp\cosh x\frac{d}{dx} + \sinh x\sqrt{-2\mathcal{H}'}. \tag{3.42}$$

When applied to the eigenvector ϕ_n, we obtain as $2\mathcal{E}_n - g^2 = -(g-n)^2$:

$$a'^{(-)}\phi_n = \quad \cosh x \frac{d\phi_n}{dx} + (g-n)\sinh x\,\phi_n = (n+\beta)\phi_{n-1}, \tag{3.43}$$

$$a'^{(+)}\phi_n = -\cosh x \frac{d\phi_n}{dx} + (g-n)\sinh x\,\phi_n = -\frac{(n+1)(n+2\beta+1)}{n+\beta+1}\phi_{n+1}. \tag{3.44}$$

We obtain the following interesting commutation relations:

$$[\mathcal{H}, a'^{(\pm)}] = \pm\big(\sqrt{-2\mathcal{H}'}\,a'^{(\pm)} + a'^{(\pm)}\sqrt{-2\mathcal{H}'}\,\big)/2, \tag{3.45}$$

$$[a'^{(-)}, a'^{(+)}] = 2\sqrt{-2\mathcal{H}'}, \qquad a'^{(-)}a'^{(+)} + a'^{(+)}a'^{(-)} = 4\mathcal{H} + 2g, \tag{3.46}$$

which look very similar to those for the $1/\sin^2 x$ potential (2.65)–(2.67). In contrast to the $1/\sin^2 x$ case, the present case has only finite dimensional representation, $n = 0, 1, \ldots, [g]'$, that is from the ground state to the highest level. There is no coherent state as the eigenvector of the annihilation operator (3.42).

3.1.4 Morse potential

This is another well-known example of exactly solvable potential with a finite number of bound states [1]:

$$\mathcal{H} \stackrel{\text{def}}{=} (p + i\mu\, e^x - ig)(p - i\mu\, e^x + ig)/2, \quad -\infty < x < \infty, \quad \mu, g > 0, \tag{3.47}$$

$$\mathcal{E}_n = n(-n/2 + g), \quad n = 0, 1, \ldots, [g]', \quad \eta(x) = e^{-x}, \tag{3.48}$$

$$\phi_n(x) = e^{-\mu\, e^x + gx} e^{-nx} L_n^{(2g-2n)}(2\mu e^x). \tag{3.49}$$

The Poisson brackets are

$$\{\mathcal{H}, e^{-x}\} = p\, e^{-x}, \quad \{\mathcal{H}, \{\mathcal{H}, e^{-x}\}\} = e^{-x} 2\mathcal{H}' + \mu g, \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} - g^2/2, \tag{3.50}$$

leading to the solution of the initial value problem:

$$e^{-x(t)} = \Big(e^{-x(0)} + \frac{\mu g}{2\mathcal{H}'_0}\Big)\cos\big[t\sqrt{-2\mathcal{H}'_0}\,\big] - p(0)e^{-x(0)}\frac{\sin\big[t\sqrt{-2\mathcal{H}'_0}\,\big]}{\sqrt{-2\mathcal{H}'_0}} - \frac{\mu g}{2\mathcal{H}'_0}. \tag{3.51}$$

It describes sinusoidal motion for bound states $\mathcal{H}'_0 < 0$ only. But the above expression is valid for the unbound motion $\mathcal{H}'_0 > 0$, too. It is easy to verify $e^{-x(t)} > 0$.

The corresponding quantum expressions are

$$[\mathcal{H}, e^{-x}] = i\, e^{-x} p - e^{-x}/2, \tag{3.52}$$

$$[\mathcal{H}, [\mathcal{H}, e^{-x}]\,] = -e^{-x}(2\mathcal{H}' + 1/4) - [\mathcal{H}, e^{-x}] - \mu(g + 1/2), \tag{3.53}$$

$$\alpha_\pm(\mathcal{H}) = -1/2 \pm \sqrt{-2\mathcal{H}'}. \tag{3.54}$$

The exact operator solution reads

$$e^{it\mathcal{H}}e^{-x}\,e^{-it\mathcal{H}}=\big(i\,e^{-x}p-e^{-x}/2\big)\frac{e^{i\alpha_+(\mathcal{H})t}-e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{-2\mathcal{H}'}}-\frac{\mu(g+1/2)}{2\mathcal{H}'+1/4}$$
$$-\big(e^{-x}(2\mathcal{H}'+1/4)+\mu(g+1/2)\big)\frac{1}{2\sqrt{-2\mathcal{H}'}}\Big(\frac{e^{i\alpha_+(\mathcal{H})t}}{\alpha_+(\mathcal{H})}-\frac{e^{i\alpha_-(\mathcal{H})t}}{\alpha_-(\mathcal{H})}\Big). \tag{3.55}$$

The annihilation and creation operators are

$$a'^{(\pm)}=a^{(\pm)}2\sqrt{-2\mathcal{H}'}=\pm e^{-x}\frac{d}{dx}+e^{-x}\sqrt{-2\mathcal{H}'}-\frac{\mu(2g+1)}{2\sqrt{-2\mathcal{H}'}\mp 1}\,. \tag{3.56}$$

When applied to the eigenvector ϕ_n, we obtain as $2\mathcal{E}_n-g^2=-(g-n)^2$:

$$a'^{(-)}\phi_n=-e^{-x}\frac{d\phi_n}{dx}+(g-n)e^{-x}\phi_n-\frac{\mu(2g+1)}{2(g-n)+1}\phi_n=\frac{4\mu^2}{2(g-n)+1}\phi_{n-1}, \tag{3.57}$$
$$a'^{(+)}\phi_n=\quad e^{-x}\frac{d\phi_n}{dx}+(g-n)e^{-x}\phi_n-\frac{\mu(2g+1)}{2(g-n)-1}\phi_n=\frac{(n+1)(2g-n)}{2(g-n)-1}\phi_{n+1}. \tag{3.58}$$

3.2 'Discrete' quantum mechanical systems

For specifying the dynamical systems belonging to the 'discrete' quantum mechanics [2, 3], we use the name of the polynomial eigenfunctions for want of universally accepted naming. The factorised Hamiltonian is given by (2.76).

3.2.1 Continuous Hahn polynomial (special case)

The factorised Hamiltonian of the continuous Hahn polynomial (special case) has a potential function V depending on two parameters:

$$V(x)=(a_1+ix)(a_2+ix),\quad -\infty<x<\infty,\quad a_1,a_2>0. \tag{3.59}$$

The eigenvalues and eigenfunctions are:

$$\mathcal{E}_n=n(n+2a_1+2a_2-1)/2,\quad n=0,1,2,\ldots, \tag{3.60}$$
$$\phi_0(x)=\sqrt{\textstyle\prod_{j=1}^2\Gamma(a_j+ix)\Gamma(a_j-ix)},\quad \eta(x)=x, \tag{3.61}$$
$$\phi_n(x)=\phi_0(x)P_n(x),\quad P_n(x)\stackrel{\text{def}}{=}p_n(x\,;a_1,a_2,a_1,a_2), \tag{3.62}$$

in which $p_n(x\,;a_1,a_2,a_1,a_2)$ is a special case of the continuous Hahn polynomial (C.11). This is a two parameter deformation of the Hermite polynomial. Thus this dynamical system is

a deformed oscillator. The classical solution shows this fact clearly:

$$\{\mathcal{H}, x\} = -\sqrt{(a_1^2 + x^2)(a_2^2 + x^2)}\,\sinh p, \quad \{\mathcal{H}, \{\mathcal{H}, x\}\} = -x\big(2\mathcal{H} + (a_1 + a_2)^2\big), \tag{3.63}$$

$$\begin{aligned} x(t) &= x(0)\cos\big[t\sqrt{2\mathcal{H}_0 + (a_1 + a_2)^2}\,\big] \\ &\quad + \sqrt{(a_1^2 + x(0)^2)(a_2^2 + x(0)^2)}\,\sinh p(0)\,\frac{\sin\big[t\sqrt{2\mathcal{H}_0 + (a_1 + a_2)^2}\,\big]}{\sqrt{2\mathcal{H}_0 + (a_1 + a_2)^2}}. \end{aligned} \tag{3.64}$$

The corresponding quantum solution is also simple:

$$[\mathcal{H}, x] = -i(T_+ - T_-)/2, \tag{3.65}$$

$$[\mathcal{H}, [\mathcal{H}, x]\,] = x(2\mathcal{H}' - 1/4) + [\mathcal{H}, x], \quad 2\mathcal{H}' \stackrel{\text{def}}{=} 2\mathcal{H} + (a_1 + a_2 - 1/2)^2, \tag{3.66}$$

$$e^{it\mathcal{H}}\,x\,e^{-it\mathcal{H}} = [\mathcal{H}, x]\frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}} + x\,\frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}}, \tag{3.67}$$

$$\alpha_\pm(\mathcal{H}) = 1/2 \pm \sqrt{2\mathcal{H}'}. \tag{3.68}$$

The annihilation and creation operators are:

$$a'^{(\pm)} = a^{(\pm)}2\sqrt{2\mathcal{H}'} = \pm[\mathcal{H}, x] \mp x\,\alpha_\mp(\mathcal{H}) = \mp i(T_+ - T_-)/2 + x(\sqrt{2\mathcal{H}'} \mp 1/2). \tag{3.69}$$

When applied to the eigenvector ϕ_n, we obtain as $2\mathcal{E}_n + (a_1 + a_2 - 1/2)^2 = (n + a_1 + a_2 - 1/2)^2$:

$$\begin{aligned} 2a'^{(-)}\phi_n &= \quad i(T_+ - T_-)\phi_n + 2x(n + a_1 + a_2)\phi_n \\ &= (n + a_1 + a_2 - 1)(n + 2a_1 - 1)(n + 2a_2 - 1)\phi_{n-1}, \end{aligned} \tag{3.70}$$

$$\begin{aligned} 2a'^{(+)}\phi_n &= -i(T_+ - T_-)\phi_n + 2x(n + a_1 + a_2 - 1)\phi_n \\ &= \frac{(n+1)(n + 2a_1 + 2a_2 - 1)}{n + a_1 + a_2}\,\phi_{n+1}. \end{aligned} \tag{3.71}$$

The similarity transformed operators act as

$$\phi_0(x)^{-1}a'^{(\pm)}\phi_0(x)\cdot P_n(x) = \Big(x(n + a_1 + a_2 - \tfrac{1}{2} \mp \tfrac{1}{2}) \mp \tfrac{i}{2}\big(V(x)e^p - V(x)^*e^{-p}\big)\Big)P_n(x). \tag{3.72}$$

The coherent state ψ and ψ' are

$$\psi(x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{\lambda^n(2a_1 + 2a_2)_{2n}}{(2a_1)_n(2a_2)_n(a_1 + a_2)_n^2}\,P_n(x), \tag{3.73}$$

$$\psi'(x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{(2\lambda)^n}{(2a_1)_n(2a_2)_n(a_1 + a_2)_n}\,P_n(x). \tag{3.74}$$

We do not know if these sums have concise expressions or not.

3.2.2 Continuous dual Hahn polynomial

The continuous dual Hahn polynomial has three parameters (a_1, a_2, a_3) and is considered as a two parameter deformation of the Laguerre polynomial $L_n^{(\alpha)}$. The factorised Hamiltonian of the continuous dual Hahn polynomial has a potential function V:

$$V(x) = \frac{\prod_{j=1}^{3}(a_j + ix)}{2ix(2ix+1)}, \quad 0 < x < \infty, \quad a_1, a_2, a_3 > 0. \tag{3.75}$$

As a dynamical system this is a deformed Calogero model, or a deformed $x^2 + 1/x^2$ potential. Like the Calogero model it has a linear spectrum and the eigenfunctions are:

$$\mathcal{E}_n = n/2, \quad n = 0, 1, 2, \ldots, \tag{3.76}$$

$$\phi_0(x) = \sqrt{\frac{\prod_{j=1}^{3}\Gamma(a_j + ix)}{\Gamma(2ix)} \frac{\prod_{j=1}^{3}\Gamma(a_j - ix)}{\Gamma(-2ix)}}, \quad \eta(x) = x^2, \tag{3.77}$$

$$\phi_n(x) = \phi_0(x) P_n(x^2), \quad P_n(\eta) \stackrel{\text{def}}{=} S_n(\eta\,; a_1, a_2, a_3), \tag{3.78}$$

in which $S_n(\eta\,; a_1, a_2, a_3)$ is the continuous dual Hahn polynomial (C.12). For deriving the classical solution, let us note that the quantum potential (3.75) has acquired quantum corrections from the classical one:

$$V_c(x) = \frac{\prod_{j=1}^{3}(a_j + ix)}{(2ix)^2}\,. \tag{3.79}$$

The classical motion is simple:

$$\{\mathcal{H}_c, x^2\} = -\frac{\sqrt{\prod_{j=1}^{3}(a_j^2 + x^2)}}{2x}\sinh p, \tag{3.80}$$

$$\{\mathcal{H}_c, \{\mathcal{H}_c, x^2\}\} = -x^2/4 + 2\mathcal{H}_c^2 + b_1\mathcal{H}_c + b_2/4, \quad b_1 \stackrel{\text{def}}{=} \sum_{1\le j\le 3} a_j\,, \quad b_2 \stackrel{\text{def}}{=} \sum_{1\le j<k\le 3} a_j a_k\,, \tag{3.81}$$

$$\begin{aligned} x^2(t) &= \big(x^2(0) - 8\mathcal{H}_{c0}^2 - 4b_1\mathcal{H}_{c0} - b_2\big)\cos[t/2] + 8\mathcal{H}_{c0}^2 + 4b_1\mathcal{H}_{c0} + b_2 \\ &\quad + \frac{\sqrt{\prod_{j=1}^{3}(a_j^2 + x^2(0))}}{x(0)}\sinh p(0)\,\sin[t/2]. \end{aligned} \tag{3.82}$$

The quantum version is almost the same:

$$[\mathcal{H}, x^2] = -ix(T_+ - T_-) - (T_+ + T_-)/2, \tag{3.83}$$

$$[\mathcal{H}, [\mathcal{H}, x^2]\,] = x^2/4 + R_{-1}(\mathcal{H}), \quad R_{-1}(\mathcal{H}) = -\big(2\mathcal{H}^2 + (b_1 - 1/2)\mathcal{H} + b_2/4\big), \tag{3.84}$$

$$e^{it\mathcal{H}}\,x^2\,e^{-it\mathcal{H}} = 2i[\mathcal{H}, x^2]\sin[t/2] + \big(x^2 + 4R_{-1}(\mathcal{H})\big)\cos[t/2] - 4R_{-1}(\mathcal{H}). \tag{3.85}$$

The annihilation and creation operators are:

$$\begin{aligned} a^{(\pm)} &= \pm[\mathcal{H}, x^2] + x^2/2 + 2R_{-1}(\mathcal{H}) \\ &= \mp ix(T_+ - T_-) \mp (T_+ + T_-)/2 + x^2/2 + 2R_{-1}(\mathcal{H}). \end{aligned} \tag{3.86}$$

When applied to the eigenvector ϕ_n, we obtain:

$$a^{(-)}\phi_n = -n \prod_{1\le j<k\le 3} (n + a_j + a_k - 1) \cdot \phi_{n-1}, \tag{3.87}$$

$$a^{(+)}\phi_n = -\phi_{n+1}. \tag{3.88}$$

The similarity transformed operators are:

$$\begin{aligned} \phi_0(x)^{-1}\ a^{(\pm)}\ \phi_0(x) &= x^2/2 - 4\tilde{\mathcal{H}}^2 - 2(b_1 - 1/2)\tilde{\mathcal{H}} - b_2/2 \\ &\qquad \mp \big((1/2 + ix)V(x)e^p + (1/2 - ix)V(x)^* e^{-p}\big), \end{aligned} \tag{3.89}$$

in which $\tilde{\mathcal{H}} = \phi_0(x)^{-1}\mathcal{H}\phi_0(x) = (V(x)e^p + V(x)^*e^{-p} - V(x) - V(x)^*)/2$ is the Hamiltonian counterpart at the polynomial level satisfying $\tilde{\mathcal{H}}P_n(x^2) = n/2\, P_n(x^2)$, see Appendix C. The coherent state is

$$\psi(x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n! \prod_{1\le j<k\le 3}(a_j + a_k)_n} P_n(x^2). \tag{3.90}$$

We do not know if this sum has a concise expression or not. The commutation relations among $\mathcal{H}$, and $a^{(\pm)}$ are more complicated than $\mathfrak{su}(1,1)$:

$$\begin{aligned} [\mathcal{H}, a^{(\pm)}] &= \pm a^{(\pm)}/2, \\ [a^{(-)}, a^{(+)}] &= 32\mathcal{H}^3 + 24(b_1 - 1/2)\mathcal{H}^2 + \big(4(b_1 - 1/2)^2 + 4b_2 + 1\big)\mathcal{H} + b_1 b_2 - a_1 a_2 a_3. \end{aligned} \tag{3.91}$$

As in the Meixner-Pollaczek case (2.94), the annihilation and creation operators for the continuous dual Hahn polynomial factorise into the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ appearing in the Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger\mathcal{A}/2$:

$$a^{(-)} = X^\dagger\mathcal{A}, \quad a^{(+)} = \mathcal{A}^\dagger X. \tag{3.92}$$

The operator X in this case reads

$$\begin{aligned} X = &-iS_+T_+ + \Big(x - iV(x - \tfrac{i}{2})^* - i\,\frac{\prod_{j=1}^3(2a_j - 1)}{8(1+x^2)}\Big)S_+ \\ &+ iS_-T_- + \Big(x + iV(x - \tfrac{i}{2})\ + i\,\frac{\prod_{j=1}^3(2a_j - 1)}{8(1+x^2)}\Big)S_-. \end{aligned} \tag{3.93}$$

These X and $X^\dagger$ compensate the shift of the parameters (a_1, a_2, a_3) caused by $\mathcal{A}^\dagger$ and $\mathcal{A}$, respectively. See Appendix **B** for more details.

3.2.3 Wilson polynomial

The Wilson polynomial has four parameters (a_1, a_2, a_3, a_4) and is considered as a three parameter deformation of the Laguerre polynomial $L_n^{(\alpha)}$. The factorised Hamiltonian (2.76) of the Wilson polynomial has a potential function V:

$$V(x) = \frac{\prod_{j=1}^{4}(a_j + ix)}{2ix(2ix+1)}, \quad 0 < x < \infty, \quad a_1, a_2, a_3, a_4 > 0. \tag{3.94}$$

As a dynamical system this is another deformation of the Calogero model, or a deformed $x^2 + 1/x^2$ potential. The spectrum is now quadratic in n and the eigenfunctions are:

$$\mathcal{E}_n = n\big(n + \textstyle\sum_{j=1}^{4} a_j - 1\big)/2, \quad n = 0, 1, 2, \ldots, \tag{3.95}$$

$$\phi_0(x) = \sqrt{\frac{\prod_{j=1}^{4}\Gamma(a_j + ix)}{\Gamma(2ix)} \frac{\prod_{j=1}^{4}\Gamma(a_j - ix)}{\Gamma(-2ix)}}, \quad \eta(x) = x^2, \tag{3.96}$$

$$\phi_n(x) = \phi_0(x) P_n(x^2), \quad P_n(\eta) \stackrel{\text{def}}{=} W_n(\eta\,; a_1, a_2, a_3, a_4), \tag{3.97}$$

in which $W_n(\eta\,; a_1, a_2, a_3, a_4)$ is the Wilson polynomial (C.13). The classical motion looks like a cross between those of the continuous Hahn and the continuous dual Hahn potentials with the classical potential V_c:

$$\{\mathcal{H}_c, x^2\} = -\frac{\sqrt{\prod_{j=1}^{4}(a_j^2 + x^2)}}{2x} \sinh p, \quad V_c(x) = \frac{\prod_{j=1}^{4}(a_j + ix)}{(2ix)^2} \tag{3.98}$$

$$\{\mathcal{H}_c, \{\mathcal{H}_c, x^2\}\} = -2x^2(\mathcal{H}_c + c_1) - R_{-1}(\mathcal{H}_c), \tag{3.99}$$

$$R_{-1}(\mathcal{H}_c) = -2(\mathcal{H}_c^2 + c_2\mathcal{H}_c + c_3), \qquad c_1 = b_1^2/8, \quad c_2 = b_2, \quad c_3 = b_1 b_3/4, \tag{3.100}$$

$$\begin{aligned} x^2(t) &= \Big(x^2(0) + \frac{R_{-1}(\mathcal{H}_{c0})}{2(\mathcal{H}_{c0} + c_1)}\Big) \cos\big[t\sqrt{2(\mathcal{H}_{c0} + c_1)}\,\big] - \frac{R_{-1}(\mathcal{H}_{c0})}{2(\mathcal{H}_{c0} + c_1)} \\ &\quad + \frac{\sqrt{\prod_{j=1}^{4}(a_j^2 + x^2(0))}}{2x(0)} \sinh p(0)\, \frac{\sin\big[t\sqrt{2(\mathcal{H}_{c0} + c_1)}\,\big]}{\sqrt{2(\mathcal{H}_{c0} + c_1)}}, \end{aligned} \tag{3.101}$$

where we use the abbreviation

$$b_1 \stackrel{\text{def}}{=} \sum_{1 \le j \le 4} a_j\,, \quad b_2 \stackrel{\text{def}}{=} \sum_{1 \le j < k \le 4} a_j a_k\,, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \le j < k < l \le 4} a_j a_k a_l\,. \tag{3.102}$$

The quantum version has almost the same form with quantum corrections in the coeffi-

cients c_1, c_2 and c_3:

$$[\mathcal{H}, x^2] = -ix(T_+ - T_-) - (T_+ + T_-)/2, \tag{3.103}$$

$$[\mathcal{H}, [\mathcal{H}, x^2]\,] = [\mathcal{H}, x^2] + 2x^2(\mathcal{H} + c_1) + R_{-1}(\mathcal{H}), \tag{3.104}$$

$$R_{-1}(\mathcal{H}) = -2(\mathcal{H}^2 + c_2\mathcal{H} + c_3), \tag{3.105}$$

$$c_1 = b_1(b_1 - 2)/8, \quad c_2 = b_2 - b_1/2, \quad c_3 = (b_1 - 2)b_3/4, \tag{3.106}$$

$$\begin{aligned} e^{it\mathcal{H}}\, x^2\, e^{-it\mathcal{H}} &= [\mathcal{H}, x^2]\frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}} - \frac{R_{-1}(\mathcal{H})}{2(\mathcal{H} + c_1)} \\ &\quad + \Big(x^2 + \frac{R_{-1}(\mathcal{H})}{2(\mathcal{H} + c_1)}\Big)\frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{2\sqrt{2\mathcal{H}'}}. \end{aligned} \tag{3.107}$$

$$\alpha_\pm(\mathcal{H}) = 1/2 \pm \sqrt{2\mathcal{H}'}, \quad 2\mathcal{H}' \stackrel{\text{def}}{=} 2\mathcal{H} + 2c_1 + 1/4. \tag{3.108}$$

The annihilation and creation operators are:

$$\begin{aligned} a'^{(\pm)} = a^{(\pm)}2\sqrt{2\mathcal{H}'} &= \pm[\mathcal{H}, x^2] \mp x^2\alpha_\mp(\mathcal{H}) + \frac{R_{-1}(\mathcal{H})}{\sqrt{2\mathcal{H}'} \pm 1/2} \\ &= \mp ix(T_+ - T_-) \mp (T_+ + T_-)/2 \mp x^2\alpha_\mp(\mathcal{H}) + \frac{R_{-1}(\mathcal{H})}{\sqrt{2\mathcal{H}'} \pm 1/2}\,. \end{aligned} \tag{3.109}$$

When applied to the eigenvector ϕ_n, we obtain as $2\mathcal{E}_n + 2c_1 + 1/4 = (2n + b_1 - 1)^2/4$:

$$a'^{(-)}\phi_n = -\frac{n\prod_{1\le j<k\le 4}(n + a_j + a_k - 1)}{(2n + b_1 - 2)(2n + b_1 - 1)}\,\phi_{n-1}\,, \tag{3.110}$$

$$a'^{(+)}\phi_n = -\frac{n + b_1 - 1}{(2n + b_1 - 1)(2n + b_1)}\,\phi_{n+1}\,. \tag{3.111}$$

The similarity transformed operators act as

$$\begin{aligned} &\phi_0(x)^{-1}a'^{(\pm)}\phi_0(x)\cdot P_n(x^2) \\ &= \Big(\pm(\mathcal{E}_n - \mathcal{E}_{n\mp1})x^2 \pm \frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm1} - \mathcal{E}_n} \mp \big((1/2 + ix)V(x)e^p + (1/2 - ix)V(x)^*e^{-p}\big)\Big)P_n(x^2). \end{aligned} \tag{3.112}$$

The coherent state ψ and ψ' are

$$\psi(x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\,\frac{(a_1 + a_2 + a_3 + a_4)_{2n}}{\prod_{1\le j<k\le 4}(a_j + a_k)_n}\,P_n(x^2), \tag{3.113}$$

$$\psi'(x) = \phi_0(x)\sum_{n=0}^{\infty}\frac{(-2\lambda)^n}{n!}\,\frac{((a_1 + a_2 + a_3 + a_4)/2)_n}{\prod_{1\le j<k\le 4}(a_j + a_k)_n}\,P_n(x^2). \tag{3.114}$$

4 Summary and Comments

Unified theory of annihilation-creation operators $a^{(\pm)}$ is developed for various exactly solvable quantum mechanical systems possessing the 'sinusoidal coordinate'. It applies to most of the degree one solvable quantum mechanical systems as well as the solvable 'discrete' quantum mechanical systems, which are also shape-invariant [2]. The eigenfunctions of the latter are described by the Askey-scheme of hypergeometric orthogonal polynomials [6]. The method provides an independent *algebraic solution* of these quantum systems. The energy spectrum is obtained à la Heisenberg and Pauli from the Heisenberg operator solution for the 'sinusoidal coordinate' η, $e^{it\mathcal{H}}\eta e^{-it\mathcal{H}}$ and the entire eigenfunctions are explicitly obtained as $\{(a^{(+)})^n\phi_0\}$, $n = 0, 1, \ldots$, in which ϕ_0 is determined by $a^{(-)}\phi_0 = 0$. Various examples are worked out in section 2 and 3. It also applies to theories with a finite number of bound states. It should be stressed that these annihilation-creation operators are *natural* ones containing the differential (difference) operators, in contradistinction to those annihilation-creation operators introduced in the algebraic theory of coherent states [9]. By a similarity transformation in terms of the ground state wavefunction ϕ_0, the Heisenberg operator solution gives the structure relation for the corresponding orthogonal polynomials [5] and the annihilation-creation operators provide their shift down-up operators. Another characteristic feature is the uniqueness. Except for the overall factor, which is intrinsically undetermined, the action $a^{(\pm)}\phi_n$ is completely determined by the Hamiltonian of the system. This means that the relative weights of the terms in $a^{(\pm)}\phi_n$ are governed by the energy spectrum. We have shown in some detail that this type of algebraic exact solvability is valid at both classical and quantum levels. This is in good contrast with shape-invariance, which is a strictly quantum notion. The necessary and sufficient condition for the existence of the 'sinusoidal coordinate' is worked out for the ordinary quantum mechanical systems in Appendix **A**. It is a good challenge to derive a corresponding result for the 'discrete' quantum mechanical systems.

Generalisation of the present formalism to multi-particle systems is highly desirable. Simplest multi-particle systems possessing the 'sinusoidal coordinate' and the corresponding Heisenberg operator solution is the Calogero systems based on any root system [11]. In fact

a more general Hamiltonian

$$\mathcal{H}=\frac{1}{2}\sum_{j=1}^{n}(p_j^2+x_j^2)+V(x),\quad \sum_{j=1}^{n}x_j\frac{\partial}{\partial x_j}V(x)=-2V(x), \tag{4.1}$$

$$[\mathcal{H},[\mathcal{H},\eta]]=4(\eta-\mathcal{H}),\quad \eta=\sum_{j=1}^{n}x_j^2, \tag{4.2}$$

of harmonic oscillators modified by a generic homogeneous degree -2 potential has the same property. The corresponding eigenfunctions are the Laguerre polynomials again [15, 11]. As is well known the annihilation-creation operators of the harmonic oscillator have a quite wide applicability in many branches of physics. We wonder if the newly found annihilation-creation operators for the other solvable quantum mechanical systems might find an equally wide range of applications.

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Appendix A: Determination of the potentials having the ‘sinusoidal coordinate’

We have seen that the existence of the ‘sinusoidal coordinate’ or the exact Heisenberg operator solution (2.14) leads to the unified definition of the annihilation-creation operators. All the examples discussed in the text share the common property of ‘shape-invariance’, thanks to which the corresponding quantum systems are exactly solvable. Here in Appendix **A** we analyse, within the context of ordinary quantum mechanics, the necessary and sufficient condition for the existence of the ‘sinusoidal coordinate’ and show that such systems constitute a sub-group of known ‘shape-invariant’ quantum mechanics. For the ‘discrete’ quantum mechanical systems, writing down corresponding conditions is easy. It would be a good challenge to provide a complete list of ‘discrete’ quantum mechanical systems admitting the ‘sinusoidal coordinate’ or the exact Heisenberg operator solution.

For a given pair $(\eta(x),\mathcal{H})$ of a coordinate function $\eta(x)$ and a Hamiltonian $\mathcal{H}$ to satisfy the exact Heisenberg operator solution (2.14) is equivalent to the condition (2.12) that the

multiple commutators of $\mathcal{H}$ with η form a closed algebra at level two

$$(\mathrm{ad}\,\mathcal{H})^2\eta(x) \equiv [\mathcal{H},[\mathcal{H},\eta(x)]\,] = \eta\, R_0(\mathcal{H}) + [\mathcal{H},\eta(x)]\, R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \tag{A.1}$$

Here the coefficients $R_0(\mathcal{H})$, $R_1(\mathcal{H})$ and $R_{-1}(\mathcal{H})$ are polynomials in the Hamiltonian $\mathcal{H}$. It should be stressed that this condition is purely algebraic and the knowledge that the eigenfunctions have the general structure (2.3) is irrelevant. The latter (2.3) is a consequence of the condition (A.1). For the ordinary quantum mechanical system with potential $V(x)$

$$\mathcal{H} = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{A.2}$$

the commutator between $\mathcal{H}$ and η reads

$$[\mathcal{H},\eta] = -\eta'\frac{d}{dx} - \frac{1}{2}\eta'', \tag{A.3}$$

$$\big(\mathrm{ad}\mathcal{H}\big)^2\eta = \eta''\frac{d^2}{dx^2} + \eta'''\frac{d}{dx} + \frac{1}{4}\eta'''' + \eta' V', \tag{A.4}$$

in which primes denote differentiation with respect to x. From (A.4) we see that the l.h.s. of (A.1) contains the derivative operator (the momentum operator) at most quadratic degree. So must be the r.h.s. since the momentum operator can come in as a part of $\mathcal{H}$ (as $p^2/2$) or as $[\mathcal{H},\eta]$, see (A.3). Then we can parametrise

$$R_0(\mathcal{H}) = r_0^{(1)}\mathcal{H} + r_0^{(0)}, \quad R_1(\mathcal{H}) = r_1, \quad R_{-1}(\mathcal{H}) = r_{-1}^{(1)}\mathcal{H} + r_{-1}^{(0)}, \tag{A.5}$$

in which $r_j^{(k)}$ are all constants. Then the coefficients of the operators $\frac{d^2}{dx^2}$, $\frac{d}{dx}$ and the function part of (A.1) give the conditions:

$$\frac{d^2\eta}{dx^2} = -\frac{1}{2}(r_0^{(1)}\eta + r_{-1}^{(1)})\,, \tag{A.6}$$

$$\frac{d^3\eta}{dx^3} = -r_1\frac{d\eta}{dx}\,, \tag{A.7}$$

$$\frac{1}{4}\frac{d^4\eta}{dx^4} + \frac{d\eta}{dx}\frac{dV}{dx} = -\frac{1}{2}r_1\frac{d^2\eta}{dx^2} + (r_0^{(1)}\eta + r_{-1}^{(1)})V + r_0^{(0)}\eta + r_{-1}^{(0)}\,. \tag{A.8}$$

The first condition (A.6) simply means that $\eta(x)$ is either a *trigonometric* or a *hyperbolic* function of x which gives an *exponential* function or a *quadratic* and *linear polynomial* in x in the degenerate limits. By comparing (A.6) and (A.7) we obtain

$$r_0^{(1)} = 2r_1. \tag{A.9}$$

Then (A.8) reduces to

$$\frac{d\eta}{dx}\frac{dV}{dx} + \frac{d^2\eta}{dx^2}\Big(2V + \frac{1}{4}r_1\Big) = r_0^{(0)}\eta + r_{-1}^{(0)},$$

which integrates easily when multiplied by $d\eta/dx$:

$$\Big(\frac{d\eta}{dx}\Big)^2\Big(V + \frac{r_1}{8}\Big) = \frac{r_0^{(0)}}{2}\eta^2 + r_{-1}^{(0)}\eta + c. \tag{A.10}$$

Here c is the constant of integration. Thus we have determined the possible form of the potential V in terms of the 'sinusoidal coordinate' $\eta(x)$ and its first derivative $d\eta/dx$, with five parameters r_1, $r_0^{(0)}$, $r_{-1}^{(1)}$, $r_{-1}^{(0)}$ and c in (A.10) and two more possible constants of integration of (A.6):

$$V(x) = \frac{1}{(\frac{d\eta}{dx})^2}\Big(\frac{r_0^{(0)}}{2}\eta^2 + r_{-1}^{(0)}\eta + c\Big) - \frac{r_1}{8}. \tag{A.11}$$

The actual number of essentially free parameters is much less, since the origin of the quadratic potential, or the location of the singularity, etc, could be freely adjusted by introducing new variable $x_{\rm new} = \alpha x + \beta$ ($\mathcal{H}_{\rm new} = \mathcal{H}/\alpha^2$). The condition that the Hamiltonian must be bounded from below imposes some constraints on the parameters. The overall additive constant is fixed uniquely when the ground state energy is required to be vanishing $\mathcal{E}_0 = 0$.

It is rather straightforward to determine all the potentials possessing the 'sinusoidal coordinate' and thus algebraically exactly solvable. They all belong to the known group of shape-invariant potentials. Except for the trivial case $V = 0$, we have

1. Rational case, $r_1 = 0$. The generic solution of (A.6) is

   $$\eta(x) = -\frac{1}{4}r_{-1}^{(1)}x^2 + c_1 x + c_2, \tag{A.12}$$

   with c_1 and c_2 being the constant of integration. Two special cases are of interest: $\eta(x) = x$ gives the harmonic oscillator and $\eta(x) = x^2$ leads to the $x^2 + 1/x^2$ potential discussed in section 3.1.1.

2. Trigonometric case, $r_1 > 0$. The generic solution of (A.6) is

   $$\eta(x) = -\frac{r_{-1}^{(1)}}{2r_1} + c_1\cos\sqrt{r_1}\,x + c_2\sin\sqrt{r_1}\,x, \tag{A.13}$$

   with c_1 and c_2 being real constants of integration due to the reality (hermiticity) of η. By rescaling and shift of the coordinate x, it reduces to the Pöschl-Teller potential discussed in section 3.1.2. The $1/\sin^2 x$ potential in section 2.1.1 and the symmetric top are obtained as degenerate cases.

3. Hyperbolic and exponential cases, $r_1 < 0$. The generic solution of (A.6) is

$$\eta(x) = -\frac{r_{-1}^{(1)}}{2r_1} + c_1 \cosh\sqrt{-r_1}\,x + c_2 \sinh\sqrt{-r_1}\,x, \tag{A.14}$$

in which the constants of integration c_1 and c_2 could be vanishing or equal $c_1 = \pm c_2$. The generic case leads to the hyperbolic Pöschl-Teller potential and the degenerate cases contain the soliton potential in section 3.1.3 and hyperbolic symmetric tops and the Morse potential in section 3.1.4, etc. We could not discuss all due to space limitation.

In all these examples, the prepotential W has also a simple expression in terms of η and $d\eta/dx$:

$$\frac{dW}{dx} = \frac{a\eta + b}{\frac{d\eta}{dx}}, \qquad a = -\sqrt{r_0^{(0)} + r_1^2/4}, \quad b = \frac{2r_{-1}^{(0)}}{2a + r_1} + \frac{r_{-1}^{(1)}}{4}. \tag{A.15}$$

Here the prepotential W is related to the ground state wave function ϕ_0 and thus to the potential V as

$$\phi_0(x) = e^{W(x)}, \quad V = \frac{1}{2}\left(\left(\frac{dW}{dx}\right)^2 + \frac{d^2W}{dx^2}\right),$$

and it plays an important role in supersymmetric (shape-invariant) quantum mechanics [1, 16].

It should be stressed that not all shape-invariant and exactly solvable potentials admit the 'sinusoidal coordinate'. Such examples are the Kepler problems in rational, spherical and hyperbolic coordinates and the Rosen-Morse potential, respectively:

$$V(x) = \frac{1}{2}\left(-\frac{2}{x} + \frac{g(g-1)}{x^2} + \frac{1}{g^2}\right), \tag{A.16}$$

$$V(x) = \frac{1}{2}\left(-2\mu \cot x + \frac{g(g-1)}{\sin^2 x} + \frac{\mu^2}{g^2} - g^2\right), \tag{A.17}$$

$$V(x) = \frac{1}{2}\left(-2\mu \coth x + \frac{g(g-1)}{\sinh^2 x} + \frac{\mu^2}{g^2} + g^2\right), \tag{A.18}$$

$$V(x) = \frac{1}{2}\left(2\mu \tanh x - \frac{g(g+1)}{\cosh^2 x} + \frac{\mu^2}{g^2} + g^2\right). \tag{A.19}$$

Their wavefunctions do not have the general structure (2.3), either.

Appendix B: Interpretation in terms of Shape Invariance

As shown in section 2, the annihilation-creation operators are completely determined once the closed relationship (2.12) among η, $[\mathcal{H}, \eta]$ and $[\mathcal{H}, [\mathcal{H}, \eta]]$ is obtained. Although it plays no active role in the determination of the annihilation-creation operators, *shape invariance* is the common property underlying all these exactly solvable Hamiltonians discussed in this paper. Therefore it is interesting as well as illuminating to understand the mechanism of the annihilation-creation operators within the framework of shape invariance. For this purpose we concentrate on the annihilation-creation operators of the Meixner-Pollaczek polynomials (2.94) and of the continuous dual Hahn polynomials (3.92), which factorise into the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ constituting the shape invariant Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger\mathcal{A}/2$. Another motivation of this Appendix is to provide a bridge between the physics of 'discrete' quantum mechanics [2] and the analysis of Askey-scheme of hypergeometric polynomials [6]. The latter focuses on the polynomial part of the eigenfunctions, whose orthogonal measure is provided by the ground state wavefunction (2.4).

Let us start with recapitulating the rudimentary facts of the shape-invariant 'discrete' quantum mechanics as developed in [2]. Knowledgeable readers may jump to the main results (B.26)–(B.29), but some intermediate results (B.7) and (B.13)–(B.17) would also be interesting in connection with the 'sinusoidal coordinate' $\eta(x)$.

A shape invariant quantum mechanical system consists of a series of isospectral Hamiltonians $\{\mathcal{H}(\boldsymbol{\lambda})\}$ parametrised by (a set of) parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots)$:

$$\mathcal{H}(\boldsymbol{\lambda}) = \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda})/2, \quad \phi_n(x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda}), \quad \mathcal{E}_n(\boldsymbol{\lambda}), \quad \text{etc.}$$

Shape invariance is tersely expressed as

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + 2\mathcal{E}_1(\boldsymbol{\lambda}), \tag{B.1}$$

where $\boldsymbol{\delta}$ is a shift of the parameter.[1] The operator $\mathcal{A}(\boldsymbol{\lambda})$ maps the eigenvectors of $\mathcal{H}(\boldsymbol{\lambda})$ to those of $\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta})$ and the other operator $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ acts in the opposite direction. In each case

[1] In the case of the Askey-Wilson polynomials, this is modified to $\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = q^{2\delta'}\mathcal{A}(\boldsymbol{\lambda}*q^{\boldsymbol{\delta}})^\dagger\mathcal{A}(\boldsymbol{\lambda}*q^{\boldsymbol{\delta}}) + 2\mathcal{E}_1(\boldsymbol{\lambda})$, where δ' is a constant and $\boldsymbol{\lambda}*q^{\boldsymbol{\delta}} = (\lambda_1 q^{\delta_1}, \lambda_2 q^{\delta_2}, \cdots)$.

studied in this paper, the parameter $\boldsymbol{\lambda}$ and the shift $\boldsymbol{\delta}$ are

$$\text{Meixner-Pollaczek} \quad : \quad \boldsymbol{\lambda} = a, \quad \boldsymbol{\delta} = 1/2, \tag{B.2}$$

$$\text{continuous Hahn} \quad : \quad \boldsymbol{\lambda} = (a_1, a_2), \quad \boldsymbol{\delta} = (1/2, 1/2), \tag{B.3}$$

$$\text{continuous dual Hahn} : \quad \boldsymbol{\lambda} = (a_1, a_2, a_3), \quad \boldsymbol{\delta} = (1/2, 1/2, 1/2), \tag{B.4}$$

$$\text{Wilson} \quad : \quad \boldsymbol{\lambda} = (a_1, a_2, a_3, a_4), \quad \boldsymbol{\delta} = (1/2, 1/2, 1/2, 1/2), \tag{B.5}$$

$$\text{Askey-Wilson} \quad : \quad \boldsymbol{\lambda} = (a_1, a_2, a_3, a_4), \quad \boldsymbol{\delta} = (1/2, 1/2, 1/2, 1/2), \quad \delta' = -1/2. \tag{B.6}$$

The ground state $\phi_0(x)$ and the orthogonal polynomial $P_n(\eta(x))$ are given in (2.83)–(2.84), (3.61)–(3.62), (3.77)–(3.78), (3.96)–(3.97) and (2.102)–(2.103). These ground states satisfy[2]

$$\phi_0(x - i/2\,; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \sqrt{V(x\,; \boldsymbol{\lambda})}\, \varphi(x - i/2)\phi_0(x\,; \boldsymbol{\lambda}), \tag{B.7}$$

where $\varphi(x) \propto \eta'(x)$ is given by

$$\text{Meixner-Pollaczek} \quad : \quad \varphi(x) = 1, \tag{B.8}$$

$$\text{continuous Hahn} \quad : \quad \varphi(x) = 1, \tag{B.9}$$

$$\text{continuous dual Hahn} : \quad \varphi(x) = 2x, \tag{B.10}$$

$$\text{Wilson} \quad : \quad \varphi(x) = 2x, \tag{B.11}$$

$$\text{Askey-Wilson} \quad : \quad \varphi(x) = -2\sin x = i(z - z^{-1}). \tag{B.12}$$

Let us consider $S_\pm(\boldsymbol{\lambda})$, $T_\pm(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$ given in (2.77)–(2.79). By using the property (B.7), we have

$$\phi_0(x\,; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1}\, S_\pm(\boldsymbol{\lambda})\ \phi_0(x\,; \boldsymbol{\lambda}) = \varphi(x)^{-1}\, e^{\pm p/2}\,, \tag{B.13}$$

$$\phi_0(x\,; \boldsymbol{\lambda})^{-1}\, S_\pm(\boldsymbol{\lambda})^\dagger\ \phi_0(x\,; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \begin{cases} V(x\,; \boldsymbol{\lambda})\, e^{p/2}\, \varphi(x) \\ V(x\,; \boldsymbol{\lambda})^*\, e^{-p/2}\, \varphi(x)\,. \end{cases} \tag{B.14}$$

(In the case of the Askey-Wilson polynomials, the following replacement is needed: $\boldsymbol{\lambda} + \boldsymbol{\delta} \Rightarrow \boldsymbol{\lambda} * q^{\boldsymbol{\delta}}$, $e^{\pm p/2} \Rightarrow q^{\pm D/2}$, $V(x\,; \boldsymbol{\lambda}) \Rightarrow V(z\,; \boldsymbol{\lambda})$. Hereafter we will omit similar remarks.) From

[2] For the Askey-Wilson polynomials, this relation reads $\phi_0(x - i\gamma/2\,; \boldsymbol{\lambda} * q^{\boldsymbol{\delta}}) = \sqrt{V(z\,; \boldsymbol{\lambda})}\, \varphi(x - i\gamma/2)\phi_0(x\,; \boldsymbol{\lambda})$, where $\gamma = \log q$.

this, we obtain

$$F(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\,\mathcal{A}(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda}) = i\,\varphi(x)^{-1}\bigl(e^{p/2}-e^{-p/2}\bigr)\,, \tag{B.15}$$

$$B(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x\,;\boldsymbol{\lambda})^{-1}\,\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\ \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = -i\bigl(V(x\,;\boldsymbol{\lambda})\,e^{p/2}-V(x\,;\boldsymbol{\lambda})^*\,e^{-p/2}\bigr)\varphi(x)\,, \tag{B.16}$$

$$\widetilde{T}_{\pm}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x\,;\boldsymbol{\lambda})^{-1}\,T_{\pm}(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda}) = \begin{cases} V(x\,;\boldsymbol{\lambda})\,e^{p} \\ V(x\,;\boldsymbol{\lambda})^*\,e^{-p}\,. \end{cases} \tag{B.17}$$

Therefore the similarity transformed Hamiltonian is

$$\begin{aligned} \widetilde{\mathcal{H}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_0(x\,;\boldsymbol{\lambda})^{-1}\,\mathcal{H}(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda}) = B(\boldsymbol{\lambda})F(\boldsymbol{\lambda})/2 \\ &= \bigl(\widetilde{T}_+(\boldsymbol{\lambda})+\widetilde{T}_-(\boldsymbol{\lambda})-V(x\,;\boldsymbol{\lambda})-V(x\,;\boldsymbol{\lambda})^*\bigr)/2\,. \end{aligned} \tag{B.18}$$

which acts on $P_n(\eta(x)\,;\boldsymbol{\lambda})$ as $\tilde{H}(\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda})$.

The forward shift operator $F(\boldsymbol{\lambda})$ and backward shift operator $B(\boldsymbol{\lambda})$ act on $P_n(\eta\,;\boldsymbol{\lambda})$ as

$$F(\boldsymbol{\lambda})P_n(\eta\,;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{B.19}$$

$$B(\boldsymbol{\lambda})P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_n(\boldsymbol{\lambda})P_{n+1}(\eta\,;\boldsymbol{\lambda}), \tag{B.20}$$

where $f_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$ are constants satisfying the relation $f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})/2 = \mathcal{E}_n(\boldsymbol{\lambda})$:

$$\text{Meixner-Pollaczek} \quad : \quad f_n(\boldsymbol{\lambda}) = 2, \quad b_n(\boldsymbol{\lambda}) = n+1, \tag{B.21}$$

$$\text{continuous Hahn} \quad : \quad f_n(\boldsymbol{\lambda}) = n+2a_1+2a_2-1, \quad b_n(\boldsymbol{\lambda}) = n+1, \tag{B.22}$$

$$\text{continuous dual Hahn} : \quad f_n(\boldsymbol{\lambda}) = -n, \quad b_n(\boldsymbol{\lambda}) = -1, \tag{B.23}$$

$$\text{Wilson} \quad : \quad f_n(\boldsymbol{\lambda}) = -n(n+\textstyle\sum_{j=1}^4 a_j-1), \quad b_n(\boldsymbol{\lambda}) = -1, \tag{B.24}$$

$$\text{Askey-Wilson} \quad : \quad f_n(\boldsymbol{\lambda}) = -q^{n/2}(q^{-n}-1)(1-a_1a_2a_3a_4q^{n-1}), \quad b_n(\boldsymbol{\lambda}) = -q^{-(n+1)/2}. \tag{B.25}$$

For the Meixner-Pollaczek and the continuous dual Hahn cases, we have seen that the annihilation-creation operators are factorised $a^{(-)} \propto X^{\dagger}\mathcal{A}$ and $a^{(+)} \propto \mathcal{A}^{\dagger}X$, (2.94), (3.92). By using (B.13)–(B.14) and (B.17), $\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}X(\boldsymbol{\lambda})\phi_0(x\,;\boldsymbol{\lambda})$ and $\phi_0(x\,;\boldsymbol{\lambda})^{-1}$ $X(\boldsymbol{\lambda})^{\dagger}\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})$ can be written down explicitly. They act on $P_n(\eta\,;\boldsymbol{\lambda})$ as for the Meixner-Pollaczek case:

$$\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\,X(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda})\cdot P_n(\eta\,;\boldsymbol{\lambda}) = 2P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{B.26}$$

$$\phi_0(x\,;\boldsymbol{\lambda})^{-1}\,X(\boldsymbol{\lambda})^{\dagger}\ \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})\cdot P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = (n+2a)P_n(\eta\,;\boldsymbol{\lambda}), \tag{B.27}$$

and for the continuous dual Hahn case:

$$\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\,X(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda})\cdot P_n(\eta\,;\boldsymbol{\lambda})=P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}),\tag{B.28}$$

$$\phi_0(x\,;\boldsymbol{\lambda})^{-1}\,X(\boldsymbol{\lambda})^\dagger\ \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})\cdot P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\prod_{1\le j<k\le 3}(n+a_j+a_k)\cdot P_n(\eta\,;\boldsymbol{\lambda}).\tag{B.29}$$

Therefore $X^\dagger$ (X) compensates the parameter shift caused by $\mathcal{A}$ ($\mathcal{A}^\dagger$), so that the effect of $a^{(-)}$ ($a^{(+)}$) is to give the polynomial with the same parameter $\boldsymbol{\lambda}$ of degree one lower (higher). This result is new.

Let us close this Appendix with a remark on the formal definition of the annihilation-creation operators used within the framework of shape-invariant quantum mechanics [9, 2]. A unitary operator $\mathcal{U}$ ($\mathcal{U}^\dagger$) is defined as a map between two orthonormal bases with neighbouring parameters, $\{\hat{\phi}_n(x;\boldsymbol{\lambda})\}$ and $\{\hat{\phi}_n(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\}$:

$$\mathcal{U}\hat{\phi}_n(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\hat{\phi}_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad \mathcal{U}^\dagger\hat{\phi}_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\hat{\phi}_n(x\,;\boldsymbol{\lambda}).\tag{B.30}$$

This allows to introduce new annihilation-creation operators in a factorised form

$$\hat{a}\stackrel{\text{def}}{=}\mathcal{U}^\dagger\mathcal{A}(\boldsymbol{\lambda}),\quad \hat{a}^\dagger=\mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{U},\tag{B.31}$$

which satisfy $\mathcal{H}=\hat{a}^\dagger\hat{a}/2=\mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda})/2$. The operator $\mathcal{U}$ is rather formal and it cannot be expressed as a differential or a difference operator. This operator $\mathcal{U}$ can be considered as unitarisation of the natural factorisation operator X discussed above.

Appendix C: Some definitions related to the hypergeometric and q-hypergeometric functions

For reader's convenience we collect several definitions related to the (q-)hypergeometric functions[6].

∘ Pochhammer symbol $(a)_n$:

$$(a)_n\stackrel{\text{def}}{=}\prod_{k=1}^{n}(a+k-1)=a(a+1)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a).\tag{C.1}$$

∘ q-Pochhammer symbol $(a\,;q)_n$:

$$(a\,;q)_n\stackrel{\text{def}}{=}\prod_{k=1}^{n}(1-aq^{k-1})=(1-a)(1-aq)\cdots(1-aq^{n-1}).\tag{C.2}$$

∘ hypergeometric series ${}_rF_s$:

$$
{}_rF_s\Bigl(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \Bigm| z\Bigr) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r)_n}{(b_1, \cdots, b_s)_n} \frac{z^n}{n!}\,, \tag{C.3}
$$

where $(a_1, \cdots, a_r)_n \stackrel{\text{def}}{=} \prod_{j=1}^{r} (a_j)_n = (a_1)_n \cdots (a_r)_n$.

∘ q-hypergeometric series (the basic hypergeometric series) ${}_r\phi_s$:

$$
{}_r\phi_s\Bigl(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \Bigm| q\,;z\Bigr) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r\,;q)_n}{(b_1, \cdots, b_s\,;q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} \frac{z^n}{(q\,;q)_n}\,, \tag{C.4}
$$

where $(a_1, \cdots, a_r\,;q)_n \stackrel{\text{def}}{=} \prod_{j=1}^{r} (a_j\,;q)_n = (a_1\,;q)_n \cdots (a_r\,;q)_n$.

∘ Bessel function $J_a(z)$:

$$
J_a(z) \stackrel{\text{def}}{=} \frac{(z/2)^a}{\Gamma(a+1)}\, {}_0F_1\Bigl(\begin{matrix} - \\ a+1 \end{matrix} \Bigm| -\frac{z^2}{4}\Bigr). \tag{C.5}
$$

∘ Hermite polynomial $H_n(x)$:

$$
H_n(x) \stackrel{\text{def}}{=} (2x)^n\, {}_2F_0\Bigl(\begin{matrix} -n/2,\ -(n-1)/2 \\ - \end{matrix} \Bigm| -\frac{1}{x^2}\Bigr). \tag{C.6}
$$

∘ Laguerre polynomial $L_n^{(\alpha)}(x)$:

$$
L_n^{(\alpha)}(x) \stackrel{\text{def}}{=} \frac{(\alpha+1)_n}{n!}\, {}_1F_1\Bigl(\begin{matrix} -n \\ \alpha+1 \end{matrix} \Bigm| x\Bigr). \tag{C.7}
$$

∘ Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$
P_n^{(\alpha,\beta)}(x) \stackrel{\text{def}}{=} \frac{(\alpha+1)_n}{n!}\, {}_2F_1\Bigl(\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \Bigm| \frac{1-x}{2}\Bigr), \tag{C.8}
$$

which satisfies $P_n^{(\beta,\alpha)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x)$.

∘ Gegenbauer polynomial $C_n^{(\lambda)}(x)$:

$$
C_n^{(\lambda)}(x) \stackrel{\text{def}}{=} \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x). \tag{C.9}
$$

∘ Meixner-Pollaczek polynomial $P_n^{(a)}(x\,;\phi)$:

$$
P_n^{(a)}(x\,;\phi) \stackrel{\text{def}}{=} \frac{(2a)_n}{n!}\, e^{in\phi}\, {}_2F_1\Bigl(\begin{matrix} -n,\ a+ix \\ 2a \end{matrix} \Bigm| 1-e^{-2i\phi}\Bigr). \tag{C.10}
$$

∘ continuous Hahn polynomial $p_n(x\,;a_1,a_2,a'_1,a'_2)$:

$$
\begin{aligned}
p_n(x\,;a_1,a_2,a'_1,a'_2) &\stackrel{\text{def}}{=} i^n \frac{(a_1+a'_1)_n(a_1+a'_2)_n}{n!} \\
&\qquad \times {}_3F_2\Bigl(\begin{matrix} -n,\ n+a_1+a_2+a'_1+a'_2-1,\ a_1+ix \\ a_1+a'_1,\ a_1+a'_2 \end{matrix} \Bigm| 1\Bigr),
\end{aligned} \tag{C.11}
$$

which is symmetric under $a_1 \leftrightarrow a_2$ and $a'_1 \leftrightarrow a'_2$ separately.

∘ continuous dual Hahn polynomial $S_n(x^2\,;a_1,a_2,a_3)$:

$$S_n(x^2\,;a_1,a_2,a_3) \stackrel{\text{def}}{=} (a_1+a_2)_n(a_1+a_3)_n\ {}_3F_2\Bigl(\genfrac{}{}{0pt}{}{-n,\,a_1+ix,\,a_1-ix}{a_1+a_2,\,a_1+a_3}\Bigm|1\Bigr), \tag{C.12}$$

which is symmetric under the permutations of (a_1,a_2,a_3).

∘ Wilson polynomial $W_n(x^2\,;a_1,a_2,a_3,a_4)$:

$$\begin{aligned}W_n(x^2\,;a_1,a_2,a_3,a_4) &\stackrel{\text{def}}{=} (a_1+a_2)_n(a_1+a_3)_n(a_1+a_4)_n\\ &\qquad\times\ {}_4F_3\Bigl(\genfrac{}{}{0pt}{}{-n,\,n+\sum_{j=1}^4 a_j-1,\,a_1+ix,\,a_1-ix}{a_1+a_2,\,a_1+a_3,\,a_1+a_4}\Bigm|1\Bigr),\end{aligned} \tag{C.13}$$

which is symmetric under the permutations of (a_1,a_2,a_3,a_4).

∘ Askey-Wilson Hahn polynomial $p_n(\cos x\,;a_1,a_2,a_3,a_4|q)$:

$$\begin{aligned}p_n(\cos x\,;a_1,a_2,a_3,a_4|q) &\stackrel{\text{def}}{=} a_1^{-n}(a_1a_2,a_1a_3,a_1a_4\,;q)_n\\ &\qquad\times\ {}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,a_1a_2a_3a_4q^{n-1},\,a_1e^{ix},\,a_1e^{-ix}}{a_1a_2,\,a_1a_3,\,a_1a_4}\Bigm|q\,;q\Bigr),\end{aligned} \tag{C.14}$$

which is symmetric under the permutations of (a_1,a_2,a_3,a_4).

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